Limits of Arbitrage and Collateral Constraints

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To my dear father Zhang Qinpu (1954–2007).

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Contents

Chapter I

Introduction

A distinguishing feature of the recent financial crises was the simultaneous disruption of intermediation in both the financial markets and the real sector. Arbitrage failure and collapse of durable goods spending occurred side by side, together triggering the huge contraction in output and employment. To understand the links between arbitrage and durable goods investment, it is essential to have a framework that captures key aspects of the dynamic frictions that cause price anomalies and concurrent deterioration in both financial and real activities. For this purpose, in this thesis we focus on studying the interaction between arbitrage trading and real economic activities in the presence of financial frictions.

The limits of arbitrage reflect the existence of financial frictions that prevent arbitrageurs to fully eliminate price anomalies. In this thesis, we model the financial frictions in terms of collateral constraints and use it to link arbitrage trading with real sector activities. As in Gromb and Vayanos (2002, 2017), we model arbitrage in the context of market segmentation. In particular, we assume that ordinary investors within each market have varying asset demands. Market segmentation thus leads to price discrepancies among similar assets. This creates arbitrage opportunities to those financial intermediaries who can trade in all markets. In our settings, collateral constraints serve as a guarantee to ensure that the market participants honor their liabilities. To prevent possible defaults, the trading parties with future liabilities are required to post sufficient collateral in advance so that they won't have the incentive to walk away from their liabilities. Furthermore, arbitrageurs are specialized financial intermediaries, who also engage in the production process or intermediation in the real sector. The collateral requirements allow them to use the capital investment to back their arbitrage trades. In this sense, collateral constraints limit intermediaries' arbitrage capacity as a function of their capital. In turn, their capital investment is also determined by the arbitrage profits.

In the first chapter, we focus on riskless arbitrage. We develop a general equilibrium framework that combines a conventional macroeconomic model with segmented financial markets. We derive analytically the dynamics of arbitrage trades, asset prices and the aggregate capital growth. Moreover, the framework also allows us to study the implications of multiple equilibria. We find that when the economy is free of unanticipated shocks, arbitrage activities can leverage up the capital investment in the real sector and boost the aggregate output. Meanwhile, the capital growth in turn supports more arbitrage trades. However, this mutually enhancing relationship also renders the economy vulnerable to potential adverse shocks. A tiny or temporary shock might cause a severe crisis by triggering a regime switch that pushes the economy towards a low-welfare equilibrium. Furthermore, our analysis on regime shifts between different equilibria also sheds light on the study of post-crisis recovery patterns.

In our second chapter, we focus on risky arbitrage. We assume that market participants do not have perfect foresight of future asset demands. Intermediaries obtain arbitrage profits by speculating on the convergence of the price spreads. However, if the future price spread instead widens up due to the volatile market demands, they would suffer losses in their trading positions. In this framework, we investigate intermediaries' trading strategies in equilibrium and the relevant impacts on durable goods investment. We find that intermediaries tend to be more conservative in trading when the arbitrage opportunity is more profitable, and more aggressive otherwise. This seemingly counterintuitive strategy is attributable to both endogenous collateral constraints and the price externality. The collateral constraint becomes endogenously looser when the arbitrage profitability is low, thus encouraging more arbitrage trades. Furthermore, we also compare the market stability under different collateral policies. We show that a tighter collateral policy helps curb the intermediaries' tendency to overinvest and stabilize the economy at the cost of market liquidity. In contrast, a looser collateral policy supports a more liquid market and less mispricing. However, it might also amplify intermediaries' losses from overinvestment and render the economy more fragile in the wake of unanticipated large demand shocks.

In the third chapter, we study the intermediation failure in the real sector associated with arbitrage trading. We develop a dynamic model that features financial accelerator effects, market liquidity co-movement and sudden self-fulfilling market collapse. We show that a tiny, temporary shock can lead to significant, persistent disruptions in both financial and real sectors. Specifically, when intermediaries are forced to unwind some arbitrage trades in a downturn, the price moves against their positions and incurs losses. Such losses further trigger a fire sale of the durable goods, suppressing the capital price and lifting the haircut rate. Accordingly, it becomes increasingly difficult for intermediaries to exploit arbitrage opportunities and to intermediate in the real sector. In this way, the loss spiral and the liquidity spiral reinforce each other. Furthermore, we model the sudden market collapse as the sunspot equilibrium in which all intermediaries become insolvent and are denied access to renewed fundings through arbitrage trades.

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Chapter II

Best Friend or Worst Enemy? – Dynamics and Multiple Equilibria with Arbitrage, Production and Collateral Constraints

This paper is a theoretical study into how arbitrage trading affects and is affected by aggregate economic activities over the business cycle. We develop a tractable framework incorporating limits of arbitrage within a conventional macroeconomic model. By featuring a wide range of collateral categories, we derive the model dynamics analytically to illustrate the role of arbitrage trading in the expansion of the aggregate economy. Also, our discussion of the multiple equilibria allows us to examine the nonlinear aspects of financial crises through regime switching, and to shed lights on the implications of post-crisis recovery.

Keywords: limit of arbitrage, financial intermediary, segmented markets, financial crises, market liquidity, multiple equilibria, collateral constraints, externality

JEL Classification: D52 D58 E44 G01 G12

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II.1 Introduction

This paper is a theoretical study into how arbitrage trading affects and is affected by the aggregate economic activities over the business cycle. In particular, for an economy featuring endogenous financial constraints and allowing for a wide range of collaterals, we examine the role of arbitrage activities on the growth and contractions in the real sector. By characterizing multiple equilibria, we ask whether a regime shift, triggered by a tiny shock, can cause severe disruptions in the financial intermediation and spill over to the production. We illustrate how the regime shift might complicate the post-crisis recovery.

For this purpose, we consider an infinite horizon economy in which households from two segmented markets have different asset demands. This gives rise to potential price discrepancies between identical assets and creates arbitrage opportunities for the intermediaries. Though intermediaries can profit from exploiting the price spreads, they also face financial frictions, i.e., collateral constraints. Such constraints arise naturally because the households cannot compel intermediaries to honor the contracts unless their asset positions are secured. In our setting, intermediaries' durable goods investment plays a dual role: both as a production factor and as collateral in arbitrage activities. Through collateral constraints, intermediaries' capital investment limits their arbitrage capacity. In turn, intermediaries finance their capital expenses with proceeds from arbitrage trading. By deriving a closed-form solution to the model dynamics, we show that intermediaries' arbitrage capacity and capital investment reinforce each other.

We start with a baseline model without asset demand shocks. We find that there is a mutually enhancing relationship between arbitrage trading and aggregate capital investment. On one hand, arbitrage activities help boost aggregate output by providing extra funding for capital investment. By intermediating asset demands across markets, arbitrageurs are essentially obtaining loans from households with zero or even negative interest rates. This effectively lowers the marginal cost of capital and thus encourages producers to expand their production scales. On the other hand, the amplified capital investment also provides collateral to support more arbitrage trades.

We then characterize and discuss the implications of multiple equilibria on arbitrage trading and aggregate output. Under certain parameterizations, we are able to derive two distinct and robust steady states. They correspond to different regimes of the economy. For example, there might exist two steady states, featuring different levels of arbitrage trades and price spreads. As more arbitrage trades support better risk sharing among households, there is a Pareto improvement if the economy shifts from the bad regime with less trading volumes to the good one. However, crises might arise if the shock happens to trigger the shift of the opposite direction.

Suppose that the economy were initially in a good regime. An abrupt, tiny shock, either in production or in asset demands, could open up the possibility for a financial crisis. For instance, the shock might at first only cause some small losses to intermediaries, which prompts them to reduce the capital investment and their arbitrage trades subsequently. However, as they cannot internalize the price impact of their trading volumes, this makes the following price spread widen up and move against their initial positions. As a result, intermediaries suffer further losses in the financial markets. However, the losses could become more alarming if agents panic and altogether move the economy towards a bad regime, featuring even wider price spreads. Since the initial trading volumes are relatively larger in the pre-shock regime, such unfavorable price movement thus might lead to more severe financial distresses or even a complete collapse of the intermediary sector. Accordingly, the impoverished intermediaries have to further cut down their capital investment and arbitrage scales, inducing sharp contractions in the aggregate output and market liquidity.

At first, it appears like crises only occur when the economy shifts from a good regime to a bad one, fueled by the pessimistic market sentiment. However, we show that similar disruptions can still arise, even if the economy converges to a good regime after the shock. In fact, independent of the market sentiment being positive or not, as long as the post-shock regime features a more divergent price spread than the initial one, intermediaries will suffer losses in their speculative positions. If their pre-shock trading volumes happen to be large, then the following financial distress can pose a serious threat to the market stability and the overall economy.

‘ Furthermore, our discussion of multiple equilibria allows us to draw implications on the post-crisis recovery. In contrast with the self-correcting feature in Gromb and Vayanos (2017), we find that the possibility of regime shifts in arbitrage trading complicates the post-shock recovery. Apart from slowing down the recapitalization process due to magnified losses, the shifts in regime might also derail the recovery in both the financial and real sectors. For example, the market liquidity supply might only rebound partially to a lower level in the new regime. Moreover, if the economy switches to a regime without any mispricings, then arbitrageurs can no longer exploit profits to further fund the capital investments. Consequently, the aggregate output and employment end up experiencing a slow and limited recovery.

To our knowledge, we are the first to set up a theoretical framework that integrates speculations across segmented markets into a conventional macroeconomic model through collateral constrained intermediaries. The paper complements the literature that focuses on asset mispricings arising from financial frictions. Shleifer and Vishny (1997) are the first to study how trading restrictions may affect arbitrageurs' capability to eliminate price anomalies. Due to asymmetric information and moral hazard, arbitrageurs bear insolvency risk under the margin requirement. Our setup of the market segmentation borrows heavily from Gromb and Vayanos (2002, 2017). They propose a dynamic model in which financially constrained arbitrageurs intermediate asset demands across segmented markets. Our main departure from these models is that we allow for a broader range of assets (as opposed to only the riskless asset) to serve as collateral in the arbitrage trading. This enables us to study the spillover effects between the financial and real sector. Also, instead of emphasizing the self-correcting dynamics as in Gromb and Vayanos (2017), our model highlights the discussion of multiple equilibria and the corresponding regime shifts, which might derail the recovery process after the shocks.

Our paper also shares features with many other models of financially constrained arbitrageurs. For example, Brunnermeier and Pedersen (2009) study the feedback loops of arbitrageurs' funding liquidity and market liquidity, and how they interact through the collateral constraints. The funding liquidity in their model captures the arbitrageurs' capability of raising debt to facilitate the arbitrage trading. Our model differs in the source and the objective of arbitrageurs' funding. The major funding comes from arbitrage profits rather than from direct borrowing, and it is used to reinvest in productive capital. Hence, the funding liquidity is reflected by the market liquidity of financial assets. In He and Krishnamurthy (2012, 2013), arbitrageurs can raise funds from less sophisticated investors to invest in a risky financial security, but this external funding has to below an exogenous ratio of their own wealth. Liu and Longstaff (2004) study the optimal arbitrage strategy of risk-averse, collateral-constrained arbitrageurs in a partial equilibrium. Xiong (2001) and Kyle and Xiong (2001) examine the impact of arbitrage capital on asset prices by analyzing the wealth effects of arbitrageurs with log utility in a continuous-time model. However, the above papers do not study the impact of arbitrage trading on the growth in the real sector.

In addition, our paper is related to macroeconomic models stressing the pecuniary externality. In our model, when intermediaries collectively reduce their arbitrage volumes, the price spread would diverge and move against their initial positions. Some recent work also underscores similar externality, as borrowers do not internalize the impact of their own leverage decisions on the systemic risk. Examples include Schmitt-Grohé and Uribe (2016); Lorenzoni (2008); Bianchi (2011); Chari and Kehoe (2016); and Brunnermeier and Sannikov (2014).

Also, this model adds to a fast growing literature stressing collateral constraints as financial frictions and their impact on asset pricing and aggregate economic activities. In general, there are two main approaches of modeling collateral constraints. The first generally assumes that there is no external enforcement to prevent potential default. Accordingly, lenders or other collateral receiving parties have to take into account of the future default possibility ex ante, based on which they impose the specific collateral requirements. In this category, the collateral constraints often limit the current borrowing or trading activities as a function of future asset prices. Examples are Kiyotaki and Moore (1997), Chien and Lustig (2010), Kübler and Schmedders (2003). The other approach assumes implicitly that there is no default in the economy. Typically, the collateral constraints are modeled with current asset prices, as it is assumed implicitly that agents would never walk away from their liabilities. In these models, agents do not have to calculate or estimate future asset prices. This allows for more flexibility in modeling shocks and uncertainty. For instance, Bianchi (2011), Mendoza (2010) and Schmitt-Grohé and Uribe (2016) model the endogenous borrowing constraints with the current relative prices. Our model belongs to the first approach and assumes limited liability in the wake of default.

Moreover, our framework falls within a general class of macro-finance models that feature multiple equilibria to characterize business cycles. Gertler and Kiyotaki (2015) study the bank runs in an economy where bank asset liquidation prices are endogenously determined and affect whether a sunspot equilibrium exists. Schmitt-Grohé and Uribe (2016) establish the existence of multiple equilibria in open-economy models in which pessimistic views about the collateral value induce the economy to slide into self-fulfilling crises.

Finally, this paper also joins the literature considering the slow recovery from the Great Recession as a transition between different regimes. Shimer (2012) and Fajgelbaum et al.

(2017) are examples discussing the existence of multiple equilibria and the transitions from one steady state to the other in the context of after-shock recovery.

The rest of the paper is organized as follows. Section 2 introduces our baseline model. Section 3 characterizes the equilibrium. Section 4 contains our discussion of the steady states. Section 5 studies the existence and statics of multiple equilibria. Section 6 extends the multiple steady state analysis to various impulse responses and recovery patterns following unexpected shocks. Section 7 concludes with a discussion of our contribution. The Appendix includes all the proofs.

II.2 Baseline Model

II.2.1 Representative Households

We consider two segmented markets, A and B, in an infinite horizon economy with one perishable good. Each market is populated with an identical continuum of representative households (hereinafter HH), which constitute a measure of l. Each household receives natural endowments in each period, which follow an exogenous process of the form

$$e_t^i = b + u_{t-1}^i \theta_t, \quad i \in \{\mathrm{A}, \mathrm{B}\},$$

where e_t^i is the endowment of one household in period t and market i, b is a constant and $u_{t-1}^i \theta_t$ is the endowment shock. In particular, $\{\theta_t\}_{t=1}^{\infty}$ is a sequence of independent identically distributed random variables, each of which follows a symmetric distribution around zero on a bounded support $\mathcal{S} = \left[-\bar{\theta}, \bar{\theta}\right]$, where $\bar{\theta} > 0$. In short, we call θ_t the shock unit. u_{t-1}^i is the shock intensity and is always revealed one period earlier. Thus, HH know their hedging demand one period in advance.

We assume that the shock intensity in two markets are identical in magnitude but opposite in direction:

$$u_t^{\mathrm{A}} = -u_t^{\mathrm{B}} > 0.$$

This setup provides a simple way to motivate the real-world price wedges between similar assets from the diverse demands in segmented markets. For expositional convenience, we further assume that $u_t \equiv u_t^{\mathrm{A}} = u > 0$ is constant for all t. Naturally, without intermediation, the consumption paths in the two markets are perfectly negatively correlated. Thus, HH from different markets have opposite hedging demands.

HH's expected utility is given by

$$\mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s \ln\left(C_{t+s}^i\right)\right], \quad i \in \{\mathrm{A},\mathrm{B}\},$$

where C_{t+s}^{i} is HH's $t+s$ consumption in market i, $i \in \{\mathrm{A},\mathrm{B}\}$, and $0 < \beta < 1$. To ensure non-negative consumptions for HH, we set $b - u\bar{\theta} > 0$.

II.2.2 Financial Assets

In addition, there exists an identical financial asset within each market, which is long-lived, in zero-net supply and pays out a dividend equal to θ_t in t. Since its dividend exactly mimics the shock unit in each period, such asset can serve as a perfect hedging instrument for HH.

Due to HH's opposite hedging demands, the asset prices, i.e. P_t^i, $i \in \{\mathrm{A},\mathrm{B}\}$, differ across markets without further intermediation. As HH in market A always experience a positive amount of shock units, i.e., $u = u_t^{\mathrm{A}} > 0$, they are eager to sell the assets to neutralize their endowment shocks. To the extent market A has negative asset demands while market B has the opposite, prices in market A tend to be lower than in market B. We define the price difference between the two markets as

$$\phi_t := P_t^{\mathrm{B}} - P_t^{\mathrm{A}}.$$

II.2.3 Intermediaries

Outside the two markets, there also exists a continuum of measure one of competitive, risk averse and infinitely lived intermediaries (hereinafter IM). Unlike HH, IM can trade financial assets simultaneously in both markets. The non-zero price discrepancies create potential arbitrage opportunities for them. They can gain immediate profit by entering long positions in the low-price market and taking short ones in the high-price region. By doing so, they also provide market liquidity to HH in both markets. To ease exposition, we assume further that IM will incur inhibitive cost if they fail to take balanced positions across markets. As a result, in order to ensure a net zero position of financial assets, IM's positions in two markets must be equal in size but differ in signs. Denote x_t as IM's

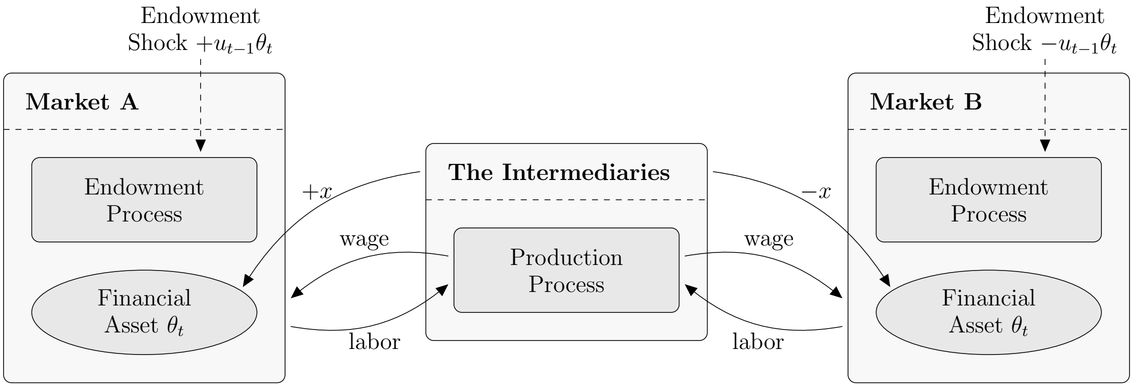

Figure II.1: The structure of the economic system.

position in market A

$$x_t := x_t^{\mathrm{A}} = -x_t^{\mathrm{B}}.$$

As in Gromb and Vayanos (2002, 2017), we use x_t as a measure of market liquidity.

Apart from intermediating asset demands between markets, IM also play a crucial role in aggregate production by investing capital as entrepreneurs. For simplicity, we assume IM have the unique capability to convert the perishable consumption goods into physical capital and vice versa, whereas HH can only convert capital into consumption. Accordingly, IM organize the production sector in the economy by providing capital input and hiring HH from both markets as labor. The production has a constant return to scale and its output function follows a Cobb-Douglas form

$$f\left(K_t, L_t\right) = aK_t^{\alpha} L_t^{\gamma},$$

where a is the total productivity parameter, α and γ are the output elasticity of capital K_t and labor L_t respectively. In addition, the capital depreciation rate is δ. IM compensate HH with a competitive wage for their labor.

IM's expected utility is given by

$$\mathbb{E}_t\left[\sum_{s=0}^{\infty} \rho^s \ln\left(C_{t+s}^{\mathrm{IM}}\right)\right],$$

where C_{t+s}^{IM} is IM's $t+s$ consumption and $0 < \rho < 1$.

II.2.4 Collateral Constraints

The only friction in this model comes from the HH's collateral requirements when they trade securities with IM. The constraints arise from the perpetual nature of the financial assets. Unlike the one-period contracts, in which asset prices collapse to zero in the next period, the positions of long-lived securities in this model remain alive and bear value in all future periods. The sequential trading of these assets thus obliges IM to first clear their previous positions before taking new ones in both markets. As will become clear, such liquidation of previous arbitrage positions usually involves an obligated payment to HH.

In order to ensure that IM honor their contracts later, HH require IM to deposit collateral up to the amount that they would not have an incentive to walk away in the next period. To be consistent with the real-world liquidation as well as the literature of limited liability (see e.g., Kübler and Schmedders (2003) and Chien and Lustig (2010)), we further assume that in case of default HH can only grab IM's depreciated capital, without being able to confiscate their capital rent. Hence, IM's collateral constraints are

$$(1-\delta)K_t - x_t\phi_{t+1} \geq 0,$$

where $-x_t\phi_{t+1} = \sum_{i\in\{\mathrm{A,B}\}} x_t^i P_{t+1}^i$ is the IM's due obligation or the value of their previous arbitrage positions.

As will be shown later, despite the following obligations, IM's trading still qualifies as arbitrage. This is because IM can essentially roll over their payment infinitely with gains from new positions.

II.2.5 Optimization Problems

Since HH draw no utility from leisure, the labor input is constant, $L_t = L = 2l$. IM's optimization problem is given by:

$$\max_{C_{t+s}^{\mathrm{IM}},x_{t+s}^i,K_{t+s}} \mathbb{E}_t\left[\sum_{s=0}^{\infty}\rho^s \ln\left(C_{t+s}^{\mathrm{IM}}\right)\right], \quad i\in\{\mathrm{A,B}\}$$

subject to

$$C_{t+s}^{\mathrm{IM}} + K_{t+s} = \underbrace{\sum_{i\in\{\mathrm{A,B}\}} x_{t+s-1}^{i} P_{t+s}^{i}}_{\text{obligations}} - \underbrace{\sum_{i\in\{\mathrm{A,B}\}} x_{t+s}^{i} P_{t+s}^{i}}_{\text{arbitrage income}} + F\left(K_{t+s-1}\right) + (1-\delta) K_{t+s-1}$$

$$\sum_{i\in\{\mathrm{A,B}\}} x_{t+s}^{i} P_{t+s+1}^{i} + (1-\delta) K_{t+s} \geq 0,$$

where $F\left(K_{t+s-1}\right) = a(1-\gamma) K_{t+s-1}^{\alpha} L^{\gamma}$.

Similar to the settings in Gromb and Vayanos (2002, 2017), IM are subject to both budget and collateral constraints. In period t, IM choose the consumption C_t^{IM}, new capital investment level K_t and the asset position in both markets x_t^i, $i \in \{\mathrm{A,B}\}$, to maximize their utility.

HH, on the other hand, cannot save or invest in physical capital. They earn their labor income from the production process. They are only subject to budget constraints. In each period t, they choose their consumption C_t^i and asset position y_t^i to solve the following problem:

$$\max_{C_{t+s}^{i}, y_{t+s}^{i}} \mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s \log\left(C_{t+s}^{i}\right) \right], \quad i \in \{\mathrm{A,B}\},$$

subject to

$$\begin{aligned} C_{t+s}^{i} &= \underbrace{\left(y_{t+s-1}^{i}\left(P_{t+s}^{i} + \theta_{t+s}\right) - y_{t+s}^{i} P_{t+s}^{i}\right)}_{\text{income from trading financial assets}} + \underbrace{a\gamma K_{t+s-1}^{\alpha} L^{\gamma-1}}_{\text{labor income}} + \underbrace{\left(b + u_{t+s-1}^{i} \theta_{t+s}\right)}_{\text{endowment}} \\ &= P_{t+s}^{i}\left(y_{t+s-1}^{i} - y_{t+s}^{i}\right) + a\gamma K_{t+s-1}^{\alpha} L^{\gamma-1} + b + \left(u_{t+s-1}^{i} + y_{t+s-1}^{i}\right)\theta_{t+s}. \end{aligned} \tag{II.1}$$

Ideally, HH would like to take a position equal to $y_t^i = -u_t^i$ in period t so that they are fully protected from endowment shocks in $t+1$.

II.2.6 Equilibrium

Given the initial capital investment K_0 and agents' asset positions x_0 and y_0^i, $i \in \{\mathrm{A,B}\}$, an equilibrium is described by the price process P_t^i, capital investment K_t, asset holdings y_t^i and x_t^i, and consumption choices C_t^{IM} and C_t^i such that

- all agents solve their optimization problems given prices;

- markets clear for financial assets, that is $y_t^i l + x_t^i = 0$.

II.3 Equilibrium Characterization

In this section, we focus on characterizing the equilibrium price differences, market liquidity, consumptions and capital investments. As we restrict IM to take balanced positions, IM's above optimization problem can be simplified as follows:

$$\max_{C_{t+s}^{\mathrm{IM}}, x_{t+s}, K_{t+s}} \mathbb{E}_t \left[\sum_{s=0}^{\infty} \rho^s \log \left(C_{t+s}^{\mathrm{IM}} \right) \right],$$

subject to

$$C_{t+s}^{\mathrm{IM}} + K_{t+s} = \underbrace{-x_{t+s-1}\phi_{t+s}}_{\text{obligation}} + \underbrace{x_{t+s}\phi_{t+s}}_{\text{arbitrage income}} + F(K_{t+s-1}) + (1-\delta)K_{t+s-1}, \tag{II.2}$$

$$- x_{t+s}\phi_{t+s+1} + (1-\delta)K_{t+s} \geq 0. \tag{II.3}$$

The Euler conditions for IM are given by

$$-\frac{\rho^{t+s}}{C_{t+s}^{\mathrm{IM}}} + \frac{\rho^{t+s+1}}{C_{t+s+1}^{\mathrm{IM}}} \left(F'\left(K_{t+s}\right) + 1 - \delta \right) + \lambda_{t+s}(1-\delta) = 0, \tag{II.4}$$

$$\frac{\rho^t \phi_{t+s}}{C_{t+s}^{\mathrm{IM}}} - \frac{\rho^{t+s+1}\phi_{t+s+1}}{C_{t+s+1}^{\mathrm{IM}}} - \lambda_{t+s}\phi_{t+s+1} = 0, \tag{II.5}$$

where $\lambda_{t+s} \geq 0$ is the Lagrange multiplier for the collateral constraint (II.3). The transversality conditions (TVC) take the form

$$\lim_{T\to\infty} \rho^{T+1} \frac{\phi_{T+1}}{C_{T+1}^{\mathrm{IM}}} x_T = 0, \tag{II.6}$$

$$\lim_{T\to\infty} \rho^{T+1} \frac{F'\left(K_T\right) + 1 - \delta}{C_{T+1}^{\mathrm{IM}}} K_T = 0. \tag{II.7}$$

Likewise, HH's first order conditions are

$$\frac{P_t^i}{C_t^i} = \beta \mathbb{E}_t \left[\frac{P_{t+1}^i + \theta_{t+1}}{C_{t+1}^i} \right], \quad i \in \{\mathrm{A}, \mathrm{B}\}. \tag{II.8}$$

Also the TVC for HH is given by

$$\lim_{T \to \infty} \beta^T \frac{P_T^i}{C_T^i} y_T^i = 0, \quad i \in \{\mathrm{A}, \mathrm{B}\}. \tag{II.9}$$

Obviously, one trivia equilibrium is $x_t = u_t, \ \forall t$. That is, from the initial period on, IM provide full liquidity in both markets and eliminate any price difference $\phi_t = 0, \ \forall t$. In this case, the required collateral is zero at all times and the collateral constraints are constantly slack. As a result, there is no arbitrage opportunity for IM and the economy thus resembles the one in neoclassical growth model. However, as will become clear later, for the settings in which alternative equilibria are possible, such a trivia equilibrium is not as robust compared to others. This is because IM are competitive, when $\phi_t = 0$ they are indifferent from taking any positions and thus they do not necessarily have incentives to commit to providing full liquidity. However, equilibrium with binding collateral constraints leaves IM no other options except sticking to the equilibrium positions. Therefore, in this instance, we rather focus on the more robust equilibrium. We regard the trivial one as a degenerate case and exclude it from the scope of our discussion.

First, we look at the equilibrium price difference.

Lemma II.1. *Define $p_t^A(\theta_t)$ and $p_t^B(\theta_t)$ to be the time t equilibrium prices in markets A and B as functions of θ_t. It follows that*

$$p_t^A(\varepsilon) = -p_t^B(-\varepsilon),$$

where $\varepsilon \in \left[-\bar{\theta}, \bar{\theta}\right]$.

Proposition II.1. *In equilibrium, the asset prices are given by*

$$P_t^A = -\frac{C_t^A}{C_t^A + C_t^B}\phi_t = -\left(\frac{1}{2} + \frac{(u - x_{t-1}/l)\,\theta_t}{2w\,(K_{t-1})}\right)\phi_t, \tag{II.10}$$

$$P_t^B = \frac{C_t^B}{C_t^A + C_t^B}\phi_t = \left(\frac{1}{2} - \frac{(u - x_{t-1}/l)\,\theta_t}{2w\,(K_{t-1})}\right)\phi_t, \tag{II.11}$$

and the price difference

$$\phi_t = \frac{2w\,(K_{t-1})}{M_t + (x_t - x_{t-1})\,/l}, \tag{II.12}$$

where

$$w\,(K_{t-1}) := a\gamma K_{t-1}^{\alpha} L^{\gamma-1} + b,$$

$$M_t := \left(\mathbb{E}_t\left[\sum_{j=1}^{\infty} \beta^j \frac{\theta_{t+j}}{C_{t+j}^B}\right]\right)^{-1}.$$

Moreover, ϕ_t is independent of the realization of shock units θ_t, $\forall t$.

Given the price difference ϕ_t, the price in i is proportional to market i's consumption share relative to all HH's consumptions. In particular, this share depends on the realization of the shock unit θ_t. Both prices are decreasing with θ_t. In contrast, the price difference ϕ_t, as a whole, is independent of the specific realization of the shock units.

Intuitively, from IM's perspective as they take a zero net position in the financial assets, their consumptions are not affected by any of the realization of dividends θ_t across time. Technically, from HH's Euler conditions, we make use of the log utility and the symmetric distribution of the shock unit θ_t. We further find that the independence of ϕ_t on θ_t also holds for the cases where HH have CRRA utility and θ_t follows a two-point distribution.

Proposition II.2. *Given the shock intensity u and the distribution of $\{\theta_t\}$, there exists a unique threshold value $\bar{\rho} > 0$ for the IM's discount factor such that:*

1. *If $\rho > \bar{\rho}$, then IM's positions $x_t = ul$ and the price spread $\phi_t = 0$ for all t.*

2. *Else, if $0 < \rho \leq \bar{\rho}$:*

- *When IM's collateral constraints are binding, then their asset positions x_t follow*

$$x_t \in (0, ul) \quad and \quad \phi_{t+1} = \frac{(1-\delta)K_t}{x_t} \tag{II.13}$$

- *When IM's collateral constraints are slack, then*

$$x_t \leq 0, \quad and \quad \frac{\phi_{t+1}}{\phi_t} = F'(K_t) + 1 - \delta. \tag{II.14}$$

Proposition II.2 shows that IM's varying degree of patience gives rise to a different form of equilibrium. As will become clear later, it also leads to distinct forms of steady states. In particular, if IM are sufficiently patient, then the markets expect them to accumulate enough collateral to eliminate all the price gaps in the long run. Such a prospect in turn prompts the markets from the very beginning to price the assets in a way that effectively clears away all the financial frictions for IM. Hence, even if IM's initial capital investment level is low, the market prices still react to allow them to provide full liquidity. As a consequence, IM are always able to overcome the collateral constraints and remove any potential mispricing resulting from market segmentation. In this instance, the economy resembles the one in the neoclassical growth model with frictionless financial markets.

On the other hand, if IM are not as patient, the market liquidity in equilibrium depends on whether IM are collateral constrained. Specifically, if IM's collateral constraints are binding, they take arbitrage positions by providing liquidity to HH in both markets. Put differently, they satisfy HH's asset demands by entering long positions in the low price market A and short positions in the high price region B. As the collateral restricts their position sizes, IM can only bring the price gaps closer, without completely eliminating them. In contrast, if IM are not collateral constrained, they will take opposite positions relative to arbitrage trading by competing liquidity with HH. This only happens in the first few periods when IM enter the financial markets with a huge amount of wealth. In order to smooth their whole consumption paths, IM tend to save and transfer resources into future periods. In this respect, IM regard the financial assets as savings instruments parallel to the physical investment, rather than an arbitrage device. By taking the opposite positions of arbitrage, in the next period IM expect to receive positive income from settlement of previous asset investment instead of paying out obligations. Meanwhile, as a saving instrument, the marginal return of taking one unit of asset in both markets should equal to that of the capital investment in equilibrium. Nevertheless, in this event, IM's trading widens up the price differences even further and exacerbates the dearth of liquidity supply in both markets.

This differs with Gromb and Vayanos (2017). They conclude that, before fully closing the price gaps, the spread decreases with arbitrageurs' wealth. Here, due to the motive of consumption smoothing, IM's excessive wealth induces more scarcity in liquidity supply and enlarges the price differences even further. The contrast mainly results from the different marginal returns of IM's saving instruments. In Gromb and Vayanos (2017), arbitrageurs can resort to a risk-free asset with constant interest rate, whereas in our model IM are facing a decreasing marginal return of their physical investment.

Next we focus on the dynamics of IM's consumption, wealth and capital investment when their collateral constraints are binding.

Proposition II.3. *With initial wealth W_0, when IM's collateral constraints are binding in t, their consumption and capital investments evolve according to*

$$C_t = (1 - \alpha\rho)W_t, \quad K_t = \alpha\rho\mu_t W_t, \tag{II.15}$$

where

$$\mu_t := \frac{\phi_{t+1}}{\phi_{t+1} - (1 - \delta)\phi_t} > 1.$$

IM's wealth dynamics is given by

$$W_{t+1} = F(K_t) + (1 - \delta)K_t - x_t\phi_{t+1} = F(K_t) = F(\alpha\rho\mu_t W_t). \tag{II.16}$$

When IM's collateral constraints keep binding, IM's remaining capital $(1-\delta)K_t$ exactly offsets their obligated cash outflow $-x_t\phi_{t+1}$. Thus IM are left with the capital rent $F(K_t)$ as wealth, which will later be allocated for their consumption and savings.

Equation (II.15) shows that IM are myopic. Independent of their income level, they allocate a fixed proportion of their wealth to savings and consumption. In particular, they save $S_t = \alpha\rho W_t$, and this proportion increases with the capital productivity and their patience level to delay consumptions. However, IM's savings are not the only source to form the new capital investment K_t. The other part comes from the entire amount of immediate arbitrage income $x_t\phi_t$. That is, IM reinvest all of their arbitrage income in the production sector. Thus,

$$K_t = \alpha\rho W_t + x_t\phi_t = \alpha\rho W_t + \frac{(1-\delta)K_t\phi_t}{\phi_{t+1}} = \alpha\rho\mu_t W_t.$$

Why would IM reinvest all in capital? From a producer's perspective, the arbitrage income $x_t\phi_t$ can be viewed as a one-period loan borrowed from HH. Likewise, the corresponding obligation $x_t\phi_{t+1}$ in the next period can be seen as the due repayment. In equilibrium, the effective interest rate of such a loan is lower than the marginal return of capital investment $F'(K_t) + 1 - \delta$. Compared with sacrificing one unit of consumption to save for one unit of physical capital, this low-rate loan offers a less painful alternative to increase capital investment. Therefore, it is in IM's favor to make full use of such external financing to leverage up their production scale and gain higher capital rent for the next period. Meanwhile, the expanded production also means higher wage/employment in both markets. In this sense, we can conclude that the arbitrage activities help boost the real economy by providing favorable credits to producers.

Proposition II.4. *Given IM's initial wealth $W_0 > 0$, an equilibrium exists in which price differences ϕ_t, IM's capital investment K_t and the market liquidity x_t are deterministic. In equilibrium, prices are non-negative in the market with negative shock intensity, i.e. market B, and non-positive in the market where the shock intensity is positive, i.e. market A.*

II.4 Steady States

When the shock intensity is constant, i.e., $u_t = u$, there are steady states in which the price difference, market liquidity and capital investment stay the same across time. In addition, there exist multiple steady states under certain conditions.

Proposition II.5. *Given the shock intensity u and the distribution of $\{\theta_t\}$, there exists a unique threshold value $\bar{\rho} > 0$ for the IM's discount factor such that:*

1. *If $\rho > \bar{\rho}$, there exists a unique steady state with slack collateral constraints for IM. In particular,*

- *IM provide full liquidity and eliminate any price gaps in the steady state, i.e.* $x_s^* = ul$ *and* $\phi_s^* = 0$.
- *IM's capital investment and wealth converge to* $K_s^* = F'^{-1}\left(\frac{1-\rho(1-\delta)}{\rho}\right)$ *and* $W_s^* = F(K_s^*) + (1-\delta)K_s^*$.

2. *Else if* $0 < \rho \leq \bar{\rho}$*, there exists steady state(s) with binding collateral constraints. Moreover, in the steady state(s),*

   - *IM's capital investment and wealth converge to*

$$K_b^* = F'^{-1}\left(\frac{\delta}{\rho}\right) > F'^{-1}\left(\frac{1-\rho(1-\delta)}{\rho}\right), \quad W_b^* = F(K_b^*).$$

   - *IM provide only partial liquidity and thus cannot eliminate the price difference completely:* $0 < x_b^* < ul$, $\phi_b^* > 0$. *In particular, the price spread converges to*

$$\phi_b^* = \frac{2\beta w(K_b^*)}{1-\beta}\mathbb{E}\left[\frac{\theta}{w(K_b^*) - (u - x_b^*/l)\,\theta}\right] > 0. \tag{II.17}$$

   - *IM's total transaction volume from arbitrage amounts to* $x_b^*\phi_b^* = (1-\delta)K_b^*$.

Intuitively, patient IM (i.e. $\rho > \bar{\rho}$) tend to save enough collateral to eliminate potential price differences. As a result, there are no unexploited arbitrage opportunities in equilibrium. Accordingly, the steady state level of IM's capital investment, consumption and wealth resemble those in the neoclassical growth model. In contrast, impatient IM wouldn't save enough capital to eliminate all arbitrage opportunities. As they are always collateral constrained, they can only afford to provide partial liquidity. In this situation, one would observe persistent price differences across segmented markets.

In a sense, the existence and exploitation of arbitrage opportunities allow IM to obtain external financing from HH. IM's arbitrage income at the steady state is essentially a nominal zero-interest loan. Since the price differences are constant, every period IM earn the same amount of arbitrage income as their due obligations. In other words, IM receive new loans in each period, which exactly offset their repayment of the previous debt. In view of this, IM's trading is indeed arbitrage in nature, as IM are able to infinitely roll over their debt with new ones in the long run. In particular, the scale of this periodical external financing equals to a fixed ratio of IM's capital $x_b^*\phi_b^* = (1-\delta)K_b^*$.

Thanks to the external financing from arbitrage opportunities, IM's production investment level in the steady states thus increases to $K_b^* = F'^{-1}(\delta/\rho)$, which is higher than that of its counterpart in the neoclassical growth model. Interestingly, because of the different possibilities of gaining external financing in the steady states, there might exist situations such that though $\rho_1 < \bar{\rho} < \rho_2$, the long run capital investment level turns out to have the converse relation $K_1^* > K_2^*$.

II.5 Multiple Equilibria

Proposition II.6. *If $\rho = \bar{\rho}$, there exists a unique steady state with a binding collateral constraint. Otherwise if $\rho \in (0, \bar{\rho})$, there exist two distinct collateral constrained steady states, denoted as $SS_1 := (x_1^*, \phi_1^*, K_1^*)$ and $SS_2 := (x_2^*, \phi_2^*, K_2^*)$, with $0 < x_1^* < x_2^* < ul$ and $\phi_1^* > \phi_2^* > 0$. In particular:*

- *In both steady states, the capital investment levels are identical, i.e. $K_1^* = K_2^* = K_b^* = F'^{-1}(\delta/\rho)$.*
- *Moreover, K_b^*, x_1^* and ϕ_2^* increase with ρ, whereas x_2^* and ϕ_1^* decrease with ρ.*

From Proposition II.5, we know that IM's steady state capital investment is independent of the particular levels of market liquidity or price spreads. Moreover, with binding collateral constraints, IM's steady state collateral amounts to not only a fixed part of the capital investment, but also their periodical obligation. IM's obligation at steady state is the product of the price spread and their arbitrage position size in each market. All else equal, the price spread decreases with IM's position size. Hence, given the obligation or their product being constant, there might exist more than one possible set of values for price spread and market liquidity.

Within the technical settings of our baseline model, for economies with impatient IM, i.e., $\rho \in (0, \bar{\rho}]$, we find two robust collateral binding steady states. The special case is $\rho = \bar{\rho}$, where essentially two steady states coincide with each other. Because IM have the same level of long-run capital investment and consumption in both steady states, they are indifferent to either of the two.

However, for HH the two distinct regimes have different welfare implications. As illustrated by a numerical example, in the left graph of Figure II.2, the two steady states differ in their level of market liquidity and the price spread. The steady state on the left, i.e., SS_1, features lower market liquidity x_1^*, larger price spread ϕ_1^* and higher price

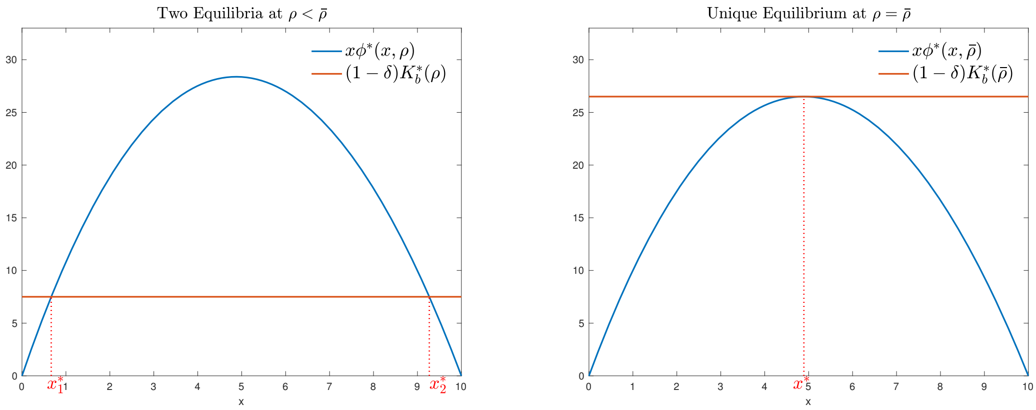

Figure II.2: $a = 4$, $b = 60$, $u = 10$, $\alpha = \gamma = 0.5$, $\delta = 0.4$, $\beta = 0.9$, $\bar{\theta} = 2$ and θ follows a two-point distribution. The horizontal line is $(1-\delta)K_b^*(\rho)$, where K_b^* is the steady state capital. The left chart is when $\rho = 0.5$, and the right one is when $\rho = \bar{\rho} = 0.94$. The blue lines are the possible products of equilibrium x and ϕ given that the steady state capital is $K_b^*(\rho)$. The interaction point(s) are the steady state position size(s) x_1^* and x_2^* (x^*).

volatilities. In contrasts, the other (SS_2) has more liquidity supply x_2^*, a narrower spread ϕ_2^* and lower price volatilities. Given that HH receive the same amount of labor income in both steady states, they prefer SS_2 over SS_1, because more market liquidity allows them to better hedge against their endowment shocks. Thus, transiting from SS_1 to SS_2 is a Pareto improvement. In what follows, we call SS_1 the unhealthy steady state/regime and refer to SS_2 as the healthy one.

Moreover, as IM become more patient, i.e., ρ increases within the range of $(0, \bar{\rho}]$, the two steady states also get "closer" to each other. The differences between the two steady state market liquidities and price spreads become smaller. The extreme case is when $\rho = \bar{\rho}$ and the healthy and unhealthy steady states are identical to one another.

Proposition II.7. *All else equal, the cutoff value $\bar{\rho}$ of IM's discount factor increases with the shock intensity u. For two otherwise identical economies with different shock intensity, e.g., $u_1 < u_2$, if IM's discount factor ρ is less than either of the cutoff values, then it follows:*

- $K_b^*[u_1] = K_b^*[u_2]$;
- $x_1^*[u_1] > x_1^*[u_2]$, $\phi_1^*[u_1] < \phi_1^*[u_2]$;
- $x_2^*[u_1] < x_2^*[u_2]$, $\phi_2^*[u_1] > \phi_2^*[u_2]$,

where $SS_i[u_j] := (K_b^*[u_j], x_i^*[u_j], \phi_i^*[u_j])$, $i \in \{1,2\}$, is the corresponding steady state in the economy with shock intensity u_j, $j \in \{1,2\}$.

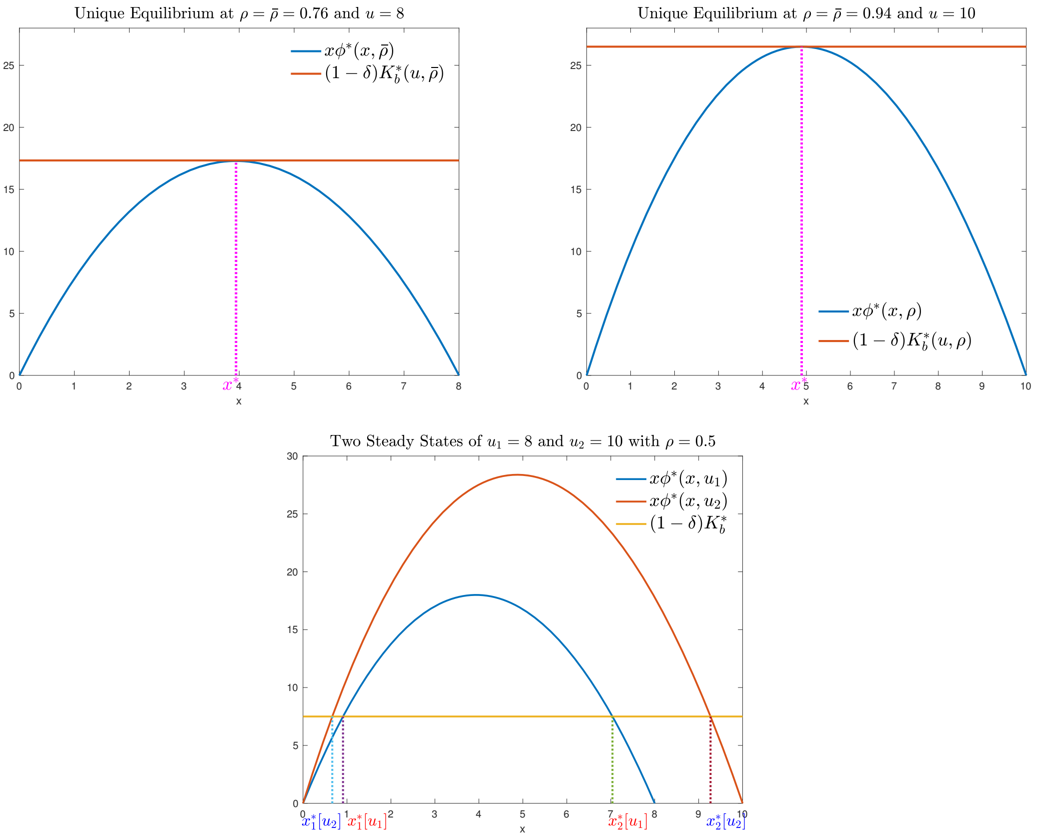

Figure II.3: $a = 8$, $b = 40$, $\alpha = \gamma = 0.5$, $\delta = 0.4$, $\beta = 0.9$, $\bar{\theta} = 2$ and θ follows a two-point distribution. The horizontal line is $(1-\delta)K_b^*$, where K_b^* is the binding steady state capital. The $x\phi^*$ lines are the possible product of equilibrium x and ϕ given that the steady state capital is K_b^* and the corresponding shock intensity. Each interaction point corresponds to a binding steady state position size x^*.

As shown in Figure II.3, all else equal, an economy with larger shock intensity allows IM of the more patient type to leverage up the aggregate production through arbitrage. On the other hand, for a given type of IM, as long as they are collateral constrained in the steady states, the level of shock intensity in the economy does not affect their aggregate capital investment. However, the gap between healthy and unhealthy steady states in terms of market liquidity and price spreads expands with increasing shock intensity. Comparably, HH enjoy better risk sharing at the healthy steady state in the economy with larger shock intensity, whereas they also suffer more from the lack of hedging capabilities if they end up in a bad steady state.

Proposition II.8. *All else equal, the cutoff value $\bar{\rho}$ of IM's discount factor decreases with the total productivity parameter a. For two otherwise identical economies with different productivity factors, e.g. $a_1 < a_2$, if IM's discount factor ρ is less than either of the cutoff values, then it follows:*

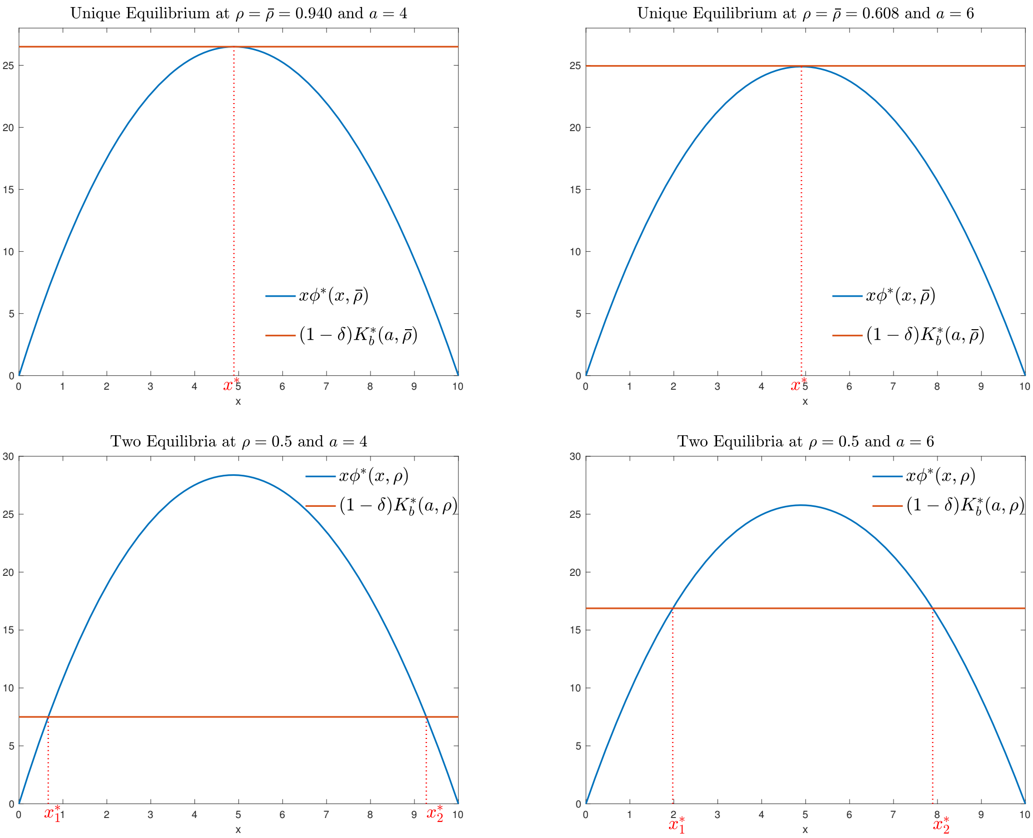

Figure II.4: $b = 60$, $u = 10$, $\alpha = \gamma = 0.5$, $\delta = 0.4$, $\beta = 0.9$, $\bar{\theta} = 2$ and θ follows a two-point distribution. The horizontal lines are $(1-\delta)K_b^*(a, \rho)$, where $K_b^*(a, \rho)$ is the binding steady state capital. The blue lines are the possible product of equilibrium x and ϕ given that the steady state capital is $K_b^*(\rho)$. Each interaction point corresponds to a binding steady state position size x^*. The upper left is when $a = 4$ and $\rho = \bar{\rho} = 0.94$. The upper right is when $a = 6$ and $\rho = \bar{\rho} = 0.608$. The lower left is when $a = 4$ and $\rho = 0.5$. The lower right is when $a = 6$ and $\rho = 0.5$.

- $K_b^*\left[a_1\right] < K_b^*\left[a_2\right]$;
- $x_1^*\left[a_1\right] < x_1^*\left[a_2\right],\ x_2^*\left[a_1\right] > x_2^*\left[a_2\right]$;
- $\phi_1^*\left[a_1\right] > \phi_1^*\left[a_2\right],\ \phi_2^*\left[a_1\right] < \phi_2^*\left[a_2\right]$;

where $SS_i\left[a_j\right] := \left(K_b^*\left[a_j\right],\ x_i^*\left[a_j\right],\ \phi_i^*\left[a_j\right]\right)$, $i \in \{1, 2\}$, *is the corresponding steady state in the economy with total productivity factor* a_j, $j \in \{1, 2\}$.

Figure II.4 compares the collateral binding steady states of economies with different total productivity factors. All else equal, higher productivities allow IM to save more capital as collateral in absolute terms. This effectively lowers the threshold value of IM's discount factor to eliminate arbitrage opportunities. Put it differently, in an economy with larger productivity factor, only IM of the relatively more impatient type are able to gain positive arbitrage profits. Meanwhile, a higher productivity factor also pushes up the steady state capital, which benefits both IM and HH with higher production income.

When comparing the binding steady states of otherwise equal economies, as shown in the lower two graphs, a higher productivity factor helps narrow down the gap between healthy and unhealthy regimes. HH in an economy with higher productivity are better off at the bad steady state in terms of higher wage and more risk sharing. However, the comparison becomes complicated at the healthy state, as it is determined by the trade-off between higher wage and lower market liquidity supply.

II.6 Implications on Recovery

In this section, we explore the implications of our model for two related issues. First, how does the economy adjust over time following sudden changes in aggregate capital or shock intensity? Second, what is the impact of potential regime shifts on the recovery of the economy? To better examine the corresponding reactions, we only consider shocks relative to the steady states of our baseline model illustrated in Proposition II.5.

II.6.1 Recovery from Sudden Losses in IM's Wealth

To study how the economy recovers from sudden losses in IM's wealth, we consider the following thought experiment. Suppose that in period t IM's wealth drops below its steady state value. This could result from, for example, an unexpected shock in their previous capital investment such as a natural disaster, or one asset failing to pay the due dividend. In the following, we investigate both the immediate and long run effects that such shocks have on aggregate output (which is also an indicator of employment), market liquidity and price spreads.

From Proposition II.6, when the collateral constraints are binding and $\rho \in (0, \bar{\rho})$, there exist two different robust equilibria. They are equally likely and their occurrences are driven by animal spirits. Thus, it is possible that an unexpected shock could trigger changes in market consensus and result in the following convergence to a different steady state. Given this possibility, we assume that if there were a negative shock hitting the IM's wealth, the market expectation of the future steady state would either remain the same or turn to a worse one. In other words, following an adverse shock the market consensus would not become more optimistic than before. This assumption is based on the belief that IM's shrinking wealth is usually associated with the prospect of reduced investment and employment (wage rate). In particular, we define a regime shift as the economy moving from one steady state to another. That is, from a healthy regime to an unhealthy one, or from an unhealthy steady state to a healthy one.

Corollary II.1. *Suppose that IM's wealth suddenly drops in period t below its steady state value and all agents keep the same expectation of the future steady state as before.*

- *The immediate effect is that the physical investment and market liquidity decrease, whereas the price discrepancy increases:*

$$K_t < K^*, \quad x_t < x^*, \quad \phi_t > \phi^*.$$

- *Following this immediate reaction, capital investment, market liquidity and price spreads revert gradually towards their pre-shock steady state levels:*

$$K_t < K_{t+1} < \cdots < K^*, \quad x_t < x_{t+1} < \cdots < x^*, \quad \phi_t > \phi_{t+1} > \cdots > \phi^*.$$

Corollary II.1 shows that when there are no regime shifts caused by changes in market sentiment, the shock reaction confirms the self-correcting pattern as described in Gromb and Vayanos (2017). In particular, the short-term drops of capital and increases of the price spread indicate that both the marginal return of production and the immediate arbitrage profitability rise above the steady state levels. These two factors serve as favorable forces to pull IM's wealth back to the pre-shock status. In the long run, the economy gradually reverts to its previous steady state.

However, this kind of recovery pattern only occurs as a special case in our model, when there is no regime switching after the shocks. When agents panic following a negative shock, the market sentiment about future economic prospect typically becomes more pessimistic. As a result, the market might expect a regime shift from the previous healthy steady state to a future unhealthy one. In this case, the resulting recovery paths will not necessarily exhibit the above ideal self-reverting feature.

Corollary II.2. *Suppose that starting from a healthy steady state $SS_h : (K_h^*,\ x_h^*,\ \phi_h^*)$, IM's wealth suddenly drops in period t below its initial value. Meanwhile, markets panic and anticipate that the economy would move towards the unhealthy steady state $SS_u : (K_u^*,\ x_u^*,\ \phi_u^*)$.*

- *If the equilibrium exists, compared to the case where there is no regime shift, then the immediate reaction is that both capital investment and market liquidity tumble more sharply, and the price spread also rises more significantly.*

$$K_t < K_{h,t} < K_u^* = K_h^*, \quad x_t < x_{u,t} < x_{h,t} < x_h^*, \quad \phi_t > \phi_{u,t} > \phi_{h,t} > \phi_h^*,$$

where $(K_{s,t},\ x_{s,t},\ \phi_{s,t})$, $s \in \{h, u\}$, *are the corresponding equilibrium level of capital investment, market liquidity and price spread in* t *after the shock, if all else equal and there is no regime shift from the pre-shock steady state* s.

- *Following the immediate reaction, IM's capital reverts gradually towards its pre-shock level. However, the price spreads and market liquidity only converge to the unhealthy steady state levels:*

$$K_t < K_{t+1} < \cdots < K_u^* = K_h^*, \quad x_t < x_{t+1} < \cdots < x_u^* < x_h^*, \quad \phi_t \to \phi_u^* > \phi_h^*.$$

Figure II.5 illustrates the equilibrium paths when there is a regime shift triggered by an unexpected negative shock. Compared to the previous benchmark recovery case, the market reacts more dramatically. Before the shock, IM carry over a larger size of arbitrage positions x_h^* from the previous period at the healthy steady state. Meanwhile, the panic market sentiment anticipates an unhealthy steady state in the future with a wider price spread ϕ_u^*. To accommodate the changing price expectation, an immediate sharp spike in the current price gap ϕ_t prevails. As IM's obligation reflects the combined two large quantities, x_h^* and ϕ_t, they are suddenly confronted with huge financial losses relative to the benchmark case.

As IM's initial loss in wealth is amplified by the huge obligation, they are forced to heavily scale back the capital investment and liquidity supply. If IM's obligation exceeds their total production income, equilibrium no longer exists. This can be interpreted as a bankrupt or bank run scenario, in which the entire production sector collapses and both financial markets enter an autarky state. Without external aids to IM, the economy risks falling into an absorbing state with permanent recession and no self-recovery capability.

In the long run, surviving IM gradually increase their capital investment and expand their liquidity supply. However, due to the heavier financial losses, it takes longer for IM

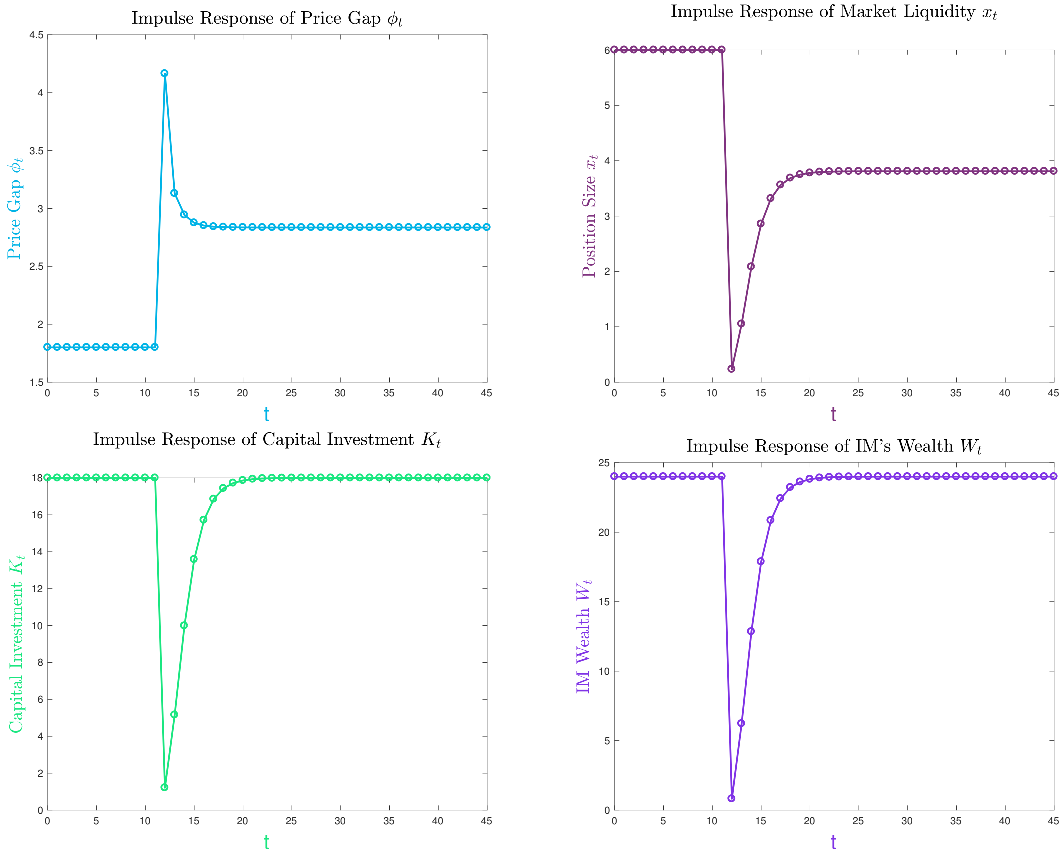

Figure II.5: Impulse response from SS_h to SS_u after a sudden loss in IM's wealth. Parameter set: $a = 8$, $b = 60$, $u = 10$, $\alpha = 0.5$, $\beta = 0.8$, $\gamma = 0.5$, $\delta = 0.4$, $\bar{\theta} = 2$, $\rho = 0.6$, $L = 2$, initial loss $\Delta W = 9$. θ_t follows a two-point distribution.

to recover their production scale, compared to the above benchmark case. What is worse, even after IM's wealth fully revives from the negative shock, HH will end up trapped in the unhealthy steady state with less protection against their endowment shocks. In this sense, the market liquidity would never completely recover.

II.6.2 Spillover Effects

So far, we have explored the amplification effects of arbitrage activities on the real economy. In the following, we will look at the corresponding spillover effects that the arbitrage activities have on the production sector. In particular, we analyze the impact of financial market shocks on IM's wealth through arbitrage activities, and how this affects the aggregate production. For this purpose, we study the reactions following sudden changes in the financial asset demands.

Response to Sudden Increase in Shock Intensity

In this section, we look into how the economy reacts to a sudden increase in the shock intensity. Specifically, we conduct the following thought experiment. Suppose that in period t the shock intensity u_t jumps abruptly to a higher level $u_1 > u$ and stays there for all future periods. For HH, this means their natural endowment processes become more volatile and accordingly their asset demands increase. By taking into account the multiple equilibria, we consider both the immediate and long run responses with different pre-shock states. We start with the case in which the collateral constraints of the pre-shock steady states are slack.

Corollary II.3. *Suppose $\rho \leq \bar{\rho}$ and at t from a certain steady state $(K^*,\ x^*,\ \phi^*)$ there is a sudden increase in the shock intensity, $u \rightarrow u_1 > u$.*

- *If the markets expect an unhealthy steady state SS_u : $(K^*,\ x^*_u,\ \phi^*_u)$ following the shock,*
  - *then immediately the price spread soars $\phi_t > \phi^*$, whereas capital investment and market liquidity slump, i.e. $K_t < K^*$ and $x_t < x^*$.*
  - *In the long run, if the equilibrium exists, the capital investment gradually recovers to its pre-shock state K^*. However, the market liquidity and price spread only converge to the unhealthy steady state levels, $x^*_u < x^*$ and $\phi^*_u > \phi^*$.*
- *Else if the markets anticipate a healthy steady state $SS_h : (K^*,\ x^*_2,\ \phi^*_h)$,*
  - *then immediately the spread shrinks $\phi_t < \phi^*$, whereas capital investment and market supply increase, i.e. $K_t > K^*$ and $x_t > x^*$.*
  - *Over time, IM's physical investment gradually falls off to its pre-shock state $K_t > K_{t+1} > \cdots > K^*$. Meanwhile, the market liquidity and the spread converge to the healthy steady state levels, $x^*_h > x^*$ and $\phi^*_h < \phi^*$.*

When the shock intensity increases to u_1, the threshold value for IM's discount factor also increases, e.g. $\bar{\rho} \rightarrow \bar{\rho}_1$. As exemplified in Figure II.6, for impatient IM, i.e. $\rho < \bar{\rho}$, the collateral constraints in the steady states stay binding before and after the shock.

The above corollary implies that independent of the starting regime, if the markets expect a bad steady state after the shock, then everyone gets worse off. Instead, if agents anticipate a healthy future regime, both HH and IM can benefit. This is because as the shock intensity increases, the gap between the healthy and unhealthy regime gets

The shock Intensity Increases from $u_1 = 8$ and $u_2 = 10$ with $\rho = 0.7$

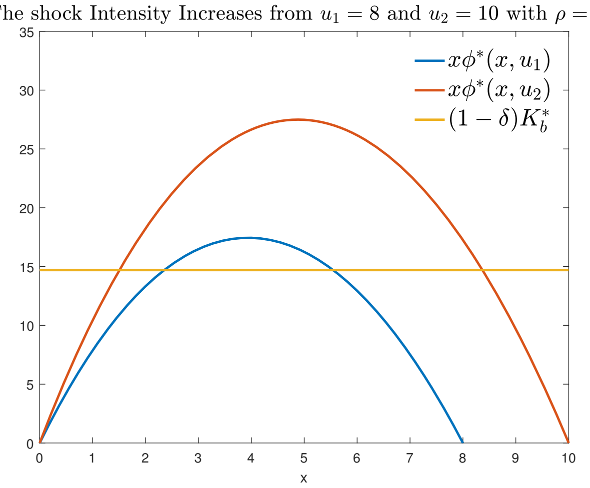

Figure II.6: $a = 8$, $b = 40$, $\alpha = \gamma = 0.5$, $\delta = 0.4$, $\rho = 0.7$, $\beta = 0.9$, $\bar{\theta} = 2$ and θ follows a two-point distribution. The horizontal line is $(1-\delta)K_b^*$, where K_b^* is the binding steady state capital. The $x\phi^*$ lines are the possible product of equilibrium x and ϕ given that the steady state capital is K_b^* and the corresponding shock intensity. Each interaction point corresponds to a binding steady state position size x^*.

more spread out. The healthy ones tend to be more "healthy" in terms of higher market liquidity, while the bad ones get "worse". When markets are optimistic about the future and anticipate a healthy regime, as shown in Figure II.8, the prospect of future narrower spread translates into a decline in current price gap. IM in effect receive an unexpected gain from their arbitrage positions, as their due obligation suddenly drops. Thus, the inflated IM's wealth immediately spills into the real economy as IM expand the capital investment. Meanwhile, HH receive higher wage in the short term and eventually enjoy better protection despite the increasing exposure.

By contrast, if markets expect an unhealthy regime with a wider price gap, disasters could happen. As shown in Figure II.7, the current price spread soars instantly to reflect the long run pessimism. IM suddenly encounter an unexpected increase in their obligations or a loss in their arbitrage positions. Such damage to IM's total income forces them to reduce the capital investment and liquidity supply. Similar to the previous discussion, in some extreme case, when the losses exceed IM's total income, there would exist no equilibrium. Before reaching the new steady state, IM suffer from less consumption and wealth, while HH are worse off from both lower wage and poorer risk sharing against higher exposure.

Corollary II.4. *Suppose in period t from a certain steady state (K^*, x^*, ϕ^*) there is a sudden increase in the shock intensity, i.e. $u \to u_1 > u$, then the threshold value for IM's discount factor jumps from $\bar{\rho}$ to $\bar{\rho}_1 > \bar{\rho}$.*

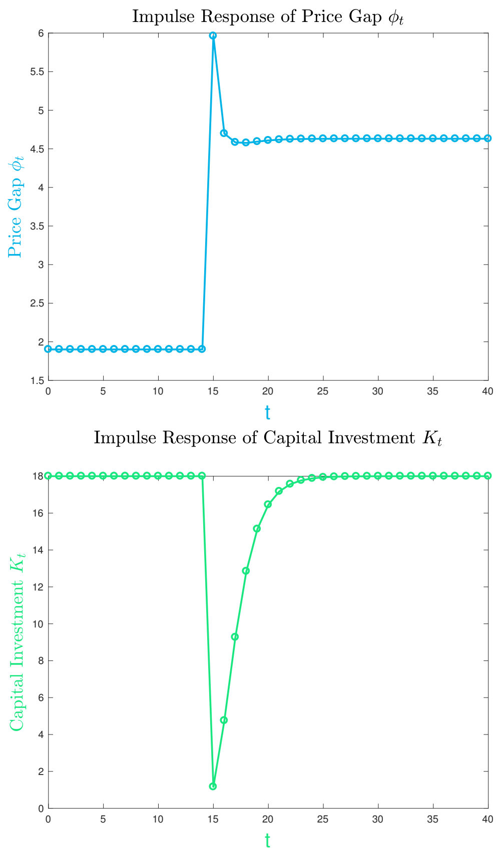

Figure II.7: Impulse responses from a previous healthy steady state to a new unhealthy steady state after the shock intensity increases from $u_1 = 9.9$ to $u_2 = 12$. Parameter set: $a = 8$, $b = 60$, $\alpha = 0.5$, $\beta = 0.8$, $\gamma = 0.5$, $\delta = 0.4$, $\bar{\theta} = 2$, $\rho = 0.6$, $L = 2$. θ_t follows a two-point distribution.

- *If $\rho > \bar{\rho}_1$, then IM immediately increase their position size u to u_1. Both the price spread and capital investment remain the same, i.e. $\phi_t = \phi_{t+1} = \cdots = \phi^* = 0$ and $K_t = K_{t+1} = \cdots = K^*$.*

- *If $\bar{\rho} < \rho \leq \bar{\rho}_1$ and $u \geq x'^*$, where x'^* is the expected steady state liquidity level after the shock, then immediately the spread increases, $\phi_t > 0$ and market liquidity drops $x_t < u$. In the long run, if equilibrium exists, the capital investment gradually increases to $K_b^* = F'^{-1}(\delta/\rho) > K^*$. IM only provide partial liquidity and the price spreads are positive.*

If IM are extremely patient, i.e. $\rho > \bar{\rho}_1$, then the steady states before and after the shock have slack collateral constraints, in which IM constantly provide full liquidity. For IM of the less patient type, i.e. $\bar{\rho} < \rho \geq \bar{\rho}_1$, the pre-shock steady state is collateral unconstrained, while the ones after the shock have binding constraints.

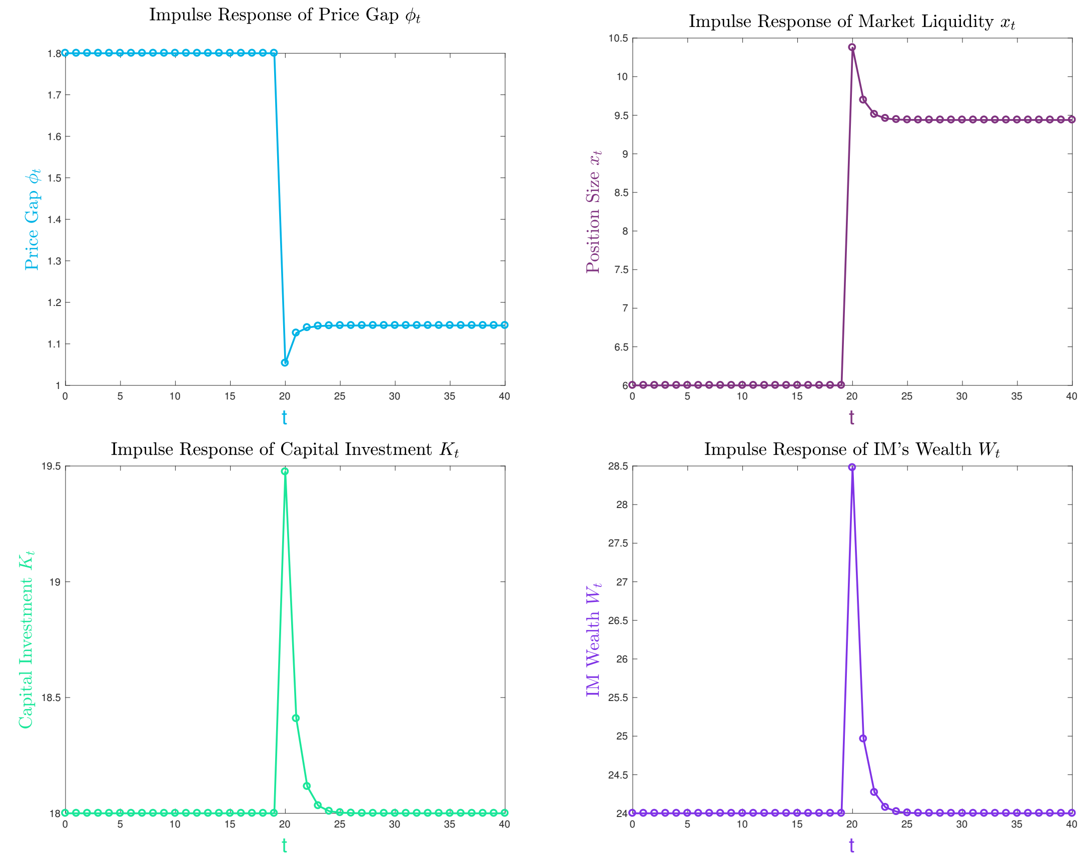

Figure II.8: Impulse responses from a previous healthy steady state to a new healthy steady state after the shock intensity increases from $u_1 = 10$ to $u_2 = 12$. Parameter set: $a = 8$, $b = 60$, $\alpha = 0.5$, $\beta = 0.8$, $\gamma = 0.5$, $\delta = 0.4$, $\bar{\theta} = 2$, $\rho = 0.6$, $L = 2$. $\{\theta_t\}$ follows a two-point distribution.

Previous discussions might give us the illusion that disasters can only happen when the markets are pessimistic and expect an unhealthy future regime. However, this corollary shows it might not be the case. Instead, even with most optimistic anticipations, similar disasters could still occur. In the case of $\bar{\rho} < \rho < \bar{\rho}_1$, as shown in Figure II.9, both after-shock steady states have positive price spreads with binding collateral constraints. Thus, even if the markets expect a healthy future regime, the price gap still jumps up immediately to accommodate the anticipated long run prices. Again, this causes IM unexpected losses in the financial markets, which will further depress the capital expenditure and liquidity supply. In case IM cannot afford to cover for the loss, no equilibrium would exist and the disastrous recession will last.

In the long term, as demonstrated in Figure II.10, surviving IM are able to exploit the arbitrage profits and increase the capital investment up to a higher level than the pre-shock state, i.e. $K_b^* > K^*$. However, HH cannot maintain full risk sharing anymore,

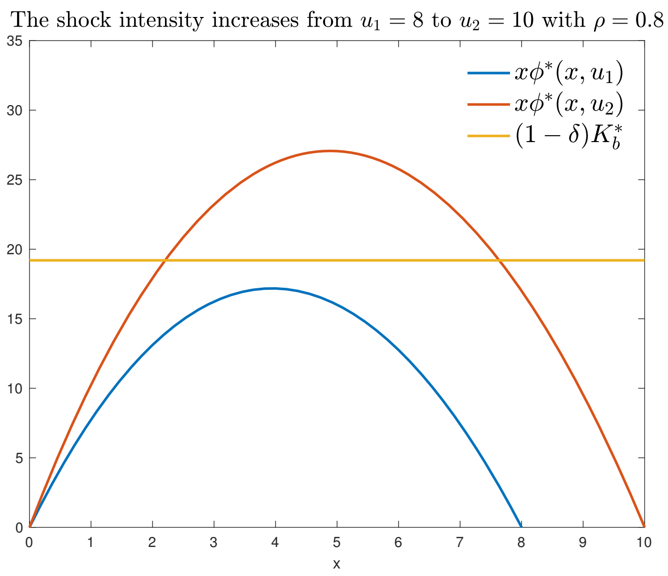

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Figure II.9: $a = 8$, $b = 40$, $\alpha = \gamma = 0.5$, $\delta = 0.4$, $\beta = 0.9$, $\bar{\theta} = 2$ and θ follows two-point distribution. The horizontal line is $(1-\delta)K_b^*$, where K_b^* is the binding steady state capital. The $x\phi^*$ lines are the possible product of equilibrium x and ϕ given that the steady state capital is K_b^* and the corresponding shock intensity. Each interaction point corresponds to a binding steady state position size x^*. With $u_1 = 8$, there exist no binding steady state.

as IM's binding collateral constraints hinder them from providing full liquidity in either of the steady states.

Response to Sudden Decrease in Shock Intensity

Likewise, we consider the following thought experiment. Suppose that in period t the shock intensity u_t plunges from its previous level u to $u_2 < u$ and sticks to it for all future periods. HH's natural endowment processes have thus become less volatile, suppressing their appetite for financial assets in both markets. In the below we study both the instantaneous and long run effects of such an unexpected drop on capital investment, market liquidity and price spreads.

We start with the case in which the collateral constraints of the ensuing steady states are slack.

Corollary II.5. *Suppose in period t at a certain steady state (K^*, x^*, ϕ^*), there is a sudden drop in the shock intensity, i.e. $u \to u_2 < u$. Then the threshold value for IM's discount factor decreases from $\bar{\rho}$ to $\bar{\rho}_2 < \bar{\rho}$.*

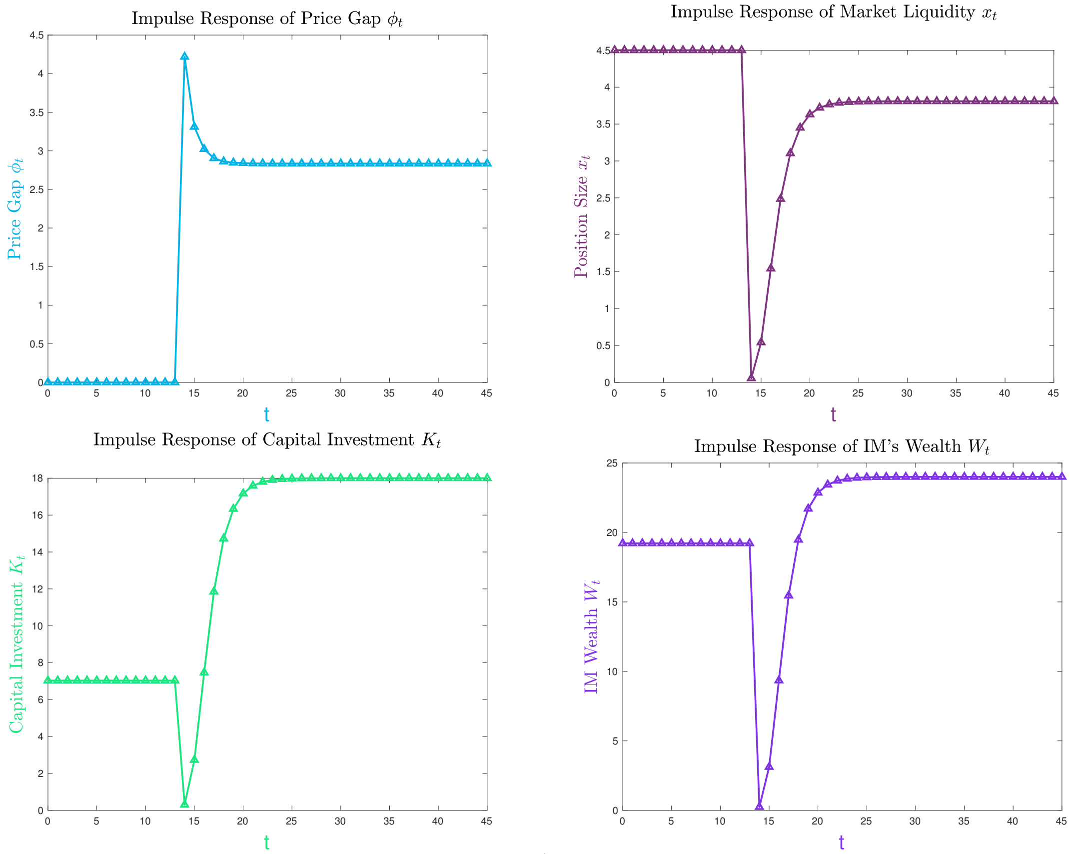

Figure II.10: Impulse responses from a previous slack steady state to an unhealthy binding steady state SS_1 after the shock intensity increases from $u_1 = 4.5$ to $u_2 = 10$. Parameter set: $a = 8$, $b = 60$, $\alpha = 0.5$, $\beta = 0.8$, $\gamma = 0.5$, $\delta = 0.4$, $\bar{\theta} = 2$, $\rho = 0.6$, $L = 2$. θ_t follows a two-point distribution.

- *If $\rho \geq \bar{\rho}$, IM immediately reduce their liquidity supply from u to u_2. Both the price spreads and capital investment remain the same, i.e. $\phi_t = \phi_{t+1} = \cdots = \phi^* = 0$ and $K_t = K_{t+1} = \cdots = K^*$.*

- *If $\bar{\rho}_2 < \rho < \bar{\rho}$, the price spread ϕ_t plunges to zero. IM provide full liquidity to both markets, i.e. $x_t = u_2$. The ensuing capital investment rises, i.e. $K_t > K^*$. In the long run, the capital falls to the corresponding steady state level as in the neoclassical growth model, i.e. $K_t > K_{t+1} > \cdots > K_s^*$.*

For patient IM, i.e. $\rho > \bar{\rho}$, their collateral requirement at the steady state before and after the shock remain unconstrained. Thus, they are able to adjust and provide full liquidity immediately after the shock.

For IM with discount factor $\bar{\rho}_2 < \rho < \bar{\rho}$, their collateral constraints are no longer binding after the shock. Thus, they can fully satisfy HH's asset demands and close out the price gaps. As a result, their obligation disappears immediately after the shock, leaving them with unexpected gains. With extra wealth, IM can temporarily raise their capital investment $K_t > K^*$ and consumption. However, as they can no longer harness the external financing to boost the production, the capital investment eventually falls off to the balanced level in the neoclassical growth model, i.e. $K_s^* < K^*$. Thus, in the long run, both IM and HH suffer from less production income compared to the pre-shock state.

Corollary II.6. *Suppose $\rho \leq \bar{\rho}_2$ and at a certain steady state $SS := (K^*,\ x^*,\ \phi^*)$ there is a sudden fall in the shock intensity in t, $u \to u_2 < u$. Denote $SS' := (K'^*,\ x'^*,\ \phi'^*)$ as the expected ensuing steady state following the shock.*

- *If SS is an unhealthy steady state,*
  - *then immediately the price spread slumps and the capital investment and market supply expand, i.e. $\phi_t < \phi'^*$, $K_t > K^*$ and $x_t > x'^*$.*
  - *In the long run, the physical investment falls to its pre-shock size, the market liquidity and the price spreads converge to the expected steady state levels, i.e. $K_t > K_{t+1} > \cdots > K^*$, $x_t > x_{t+1} > \cdots > x'^*$ and $\phi_t < \phi_{t+1} < \cdots < \phi'^*$.*
- *If SS is a healthy steady state,*
  - *then immediately the price spread surges, the capital investment and market liquidity drop, i.e. $\phi_t > \phi'^* > \phi^*$, $K_t < K^*$ and $x_t < x'^* < x^*$.*
  - *In the long run, if an equilibrium exists, the capital investment gradually grows to its pre-shock scale $K^* = K'^*$. The market liquidity and price spreads converge to the new steady state levels, i.e. x'^* and ϕ'^*.*

This corollary indicates that regardless of the prospected future steady state, if the economy is at an unhealthy (healthy) pre-shock regime, then everyone will become better (worse) off following the shock. It follows that when the shock intensity drops, the discrepancy between the good and bad steady state, in terms of market liquidity and price gaps, narrows down. When the pre-shock economy is already at an unhealthy state with a wide price gap, then the expected spread in any future regime following the intensity drop turns out to be smaller. This gives rise to an immediate decline in the price difference, which further helps IM ease up the budget from reduced obligations. As shown in Figure II.11, even moving to an unhealthy regime, the resulting capital investment and liquidity supply still rise up, which benefit both HH and IM. In the long run, as the liquidity stays at a higher level, HH enjoy better risk sharing.

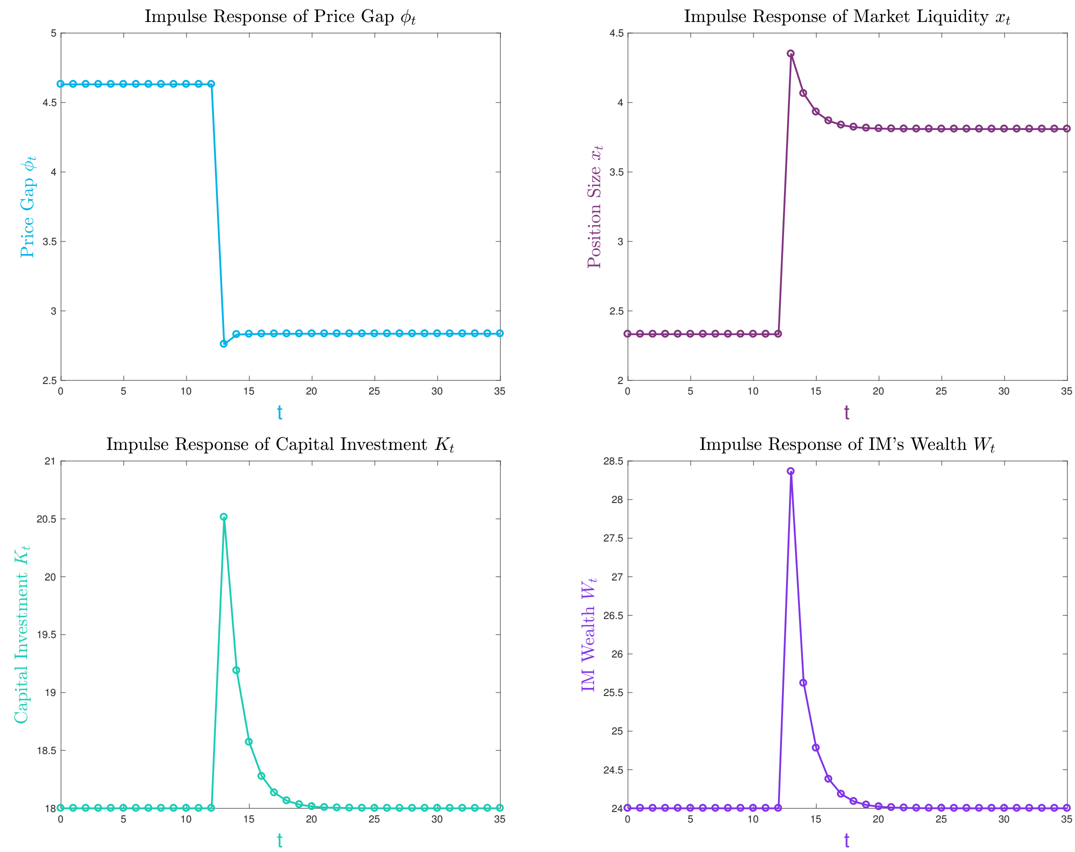

Figure II.11: Impulse responses from a previous unhealthy steady state to a new unhealthy steady state after the shock intensity drops from $u_1 = 12$ to $u_2 = 10$. Parameter set: $a = 8$, $b = 60$, $\alpha = 0.5$, $\beta = 0.8$, $\gamma = 0.5$, $\delta = 0.4$, $\bar{\theta} = 2$, $\rho = 0.6$, $L = 2$. θ_t follows a two-point distribution.

Unfortunately, there will always be a financial crisis and following recessions, if the pre-shock state is a healthy one. This holds even when the market sentiment is highly optimistic and anticipates a healthy forthcoming steady state after the shock. As illustrated in Figure II.12, even at the healthy anticipated steady state, the price spread is wider. This prompts an instant jump in the current price gap, which hurts IM's wealth through excessive obligations. With a shrinking budget, IM have to scale back the production investment and asset supply. In case IM's wealth drops below zero, no equilibrium would exist. During the recovery process, both IM and HH suffer from less production income. Besides, HH are exposed to more volatile endowment processes after the shock, despite that the shock intensity has fallen.

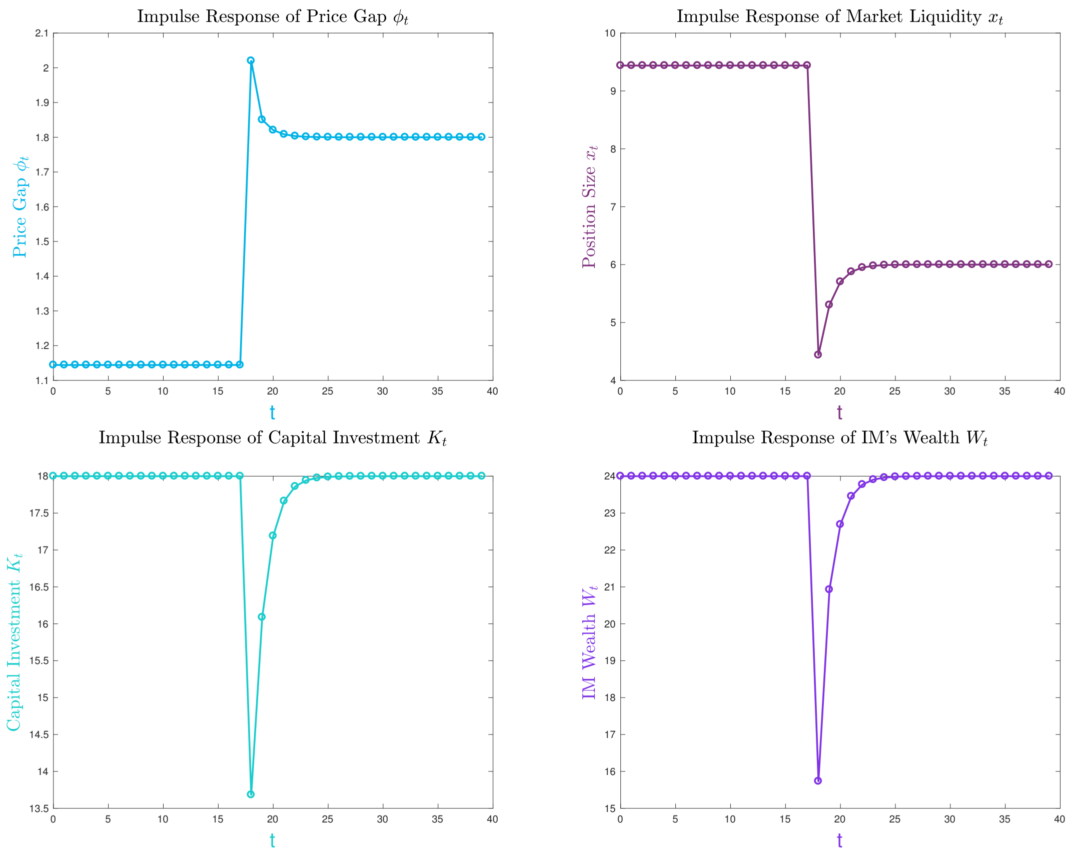

Figure II.12: Impulse responses from a previous healthy steady state to a new healthy steady state after the shock intensity drops from $u_1 = 12$ to $u_2 = 10$. Parameter set: $a = 8$, $b = 60$, $\alpha = 0.5$, $\beta = 0.8$, $\gamma = 0.5$, $\delta = 0.4$, $\bar{\theta} = 2$, $\rho = 0.6$, $L = 2$. θ_t follows a two-point distribution.

II.7 Conclusion

In this paper, we develop a highly tractable general equilibrium model of collateral constrained arbitrage and associate it with aggregate economic activities. With this simple model, we mainly address two issues: how financial frictions affect the limits of arbitrage from the asset pricing point of view; and how the arbitrage trading affects and is affected by the real activities.

At the first level, we illustrate that persistent price gaps between identical assets arise from a lack of market liquidity. The scarcity occurs in two distinct cases. One situation is when arbitrageurs are collateral constrained and they cannot afford to provide full liquidity to the markets. In this event, the price spreads are negatively correlated with arbitrageurs' collateral holdings or wealth. The other scenario is rather rare. It only happens when intermediaries carry huge initial wealth and are eager to transfer current excessive resources to the future periods. Out of consumption smoothing motives, they would rather take the opposite positions relative to arbitrage trading. By doing this, they are essentially competing for liquidity with other investors, instead of providing it to the markets. In fact, they contribute to exacerbating rather than easing the scarcity. As a result, the price discrepancies widen even further and bear positive correlation with intermediaries' wealth.

From the macroeconomic point of view, we explore both the constructive and detrimental effects of arbitrage on the real economy. We show that in normal times, arbitrage activities essentially bring low-cost external funds to the real sector. This helps expand the investment scale and boost the aggregate output. However, in the wake of sudden shocks, arbitrage trading also amplifies intermediaries' losses through significant price movements arising from the regime shift. The resulting financial distress is then fed back to the real economy and causes further recessions. Finally, the possibility of regime switches complicates the post-crisis recovery process. This follows from the chances that the ensuing long run steady state does not necessarily coincide with the pre-shock one.

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II.A Proofs

II.A.1 Proof of Lemma 1

Proof. We prove the lemma through backward induction. Suppose at $s = t + T$,

$$p^{\mathrm{A}}_{t+T}(\varepsilon) = -p^{\mathrm{B}}_{t+T}(-\varepsilon).$$

Define $c^i_s(\theta_s)$ to be the equilibrium consumption of HH in i at time s as a function of θ_s, for $i \in \{\mathrm{A}, \mathrm{B}\}$. Since $\{\theta_s\}$ follows a symmetric distribution around zero, then it must hold that

$$\frac{p^{\mathrm{A}}_{t+T}(\varepsilon)}{c^{\mathrm{A}}_{t+T}(\varepsilon)} = -\frac{p^{\mathrm{B}}_{t+T}(-\varepsilon)}{c^{\mathrm{B}}_{t+T}(-\varepsilon)}.$$

Thus,

$$\beta^T \mathbb{E}\left[\frac{P^{\mathrm{A}}_{t+T}}{C^{\mathrm{A}}_{t+T}}\right] = -\beta^T \mathbb{E}\left[\frac{P^{\mathrm{B}}_{t+T}}{C^{\mathrm{B}}_{t+T}}\right].$$

as $c^{\mathrm{A}}_{t+T}(\varepsilon) = c^{\mathrm{B}}_{t+T}(-\varepsilon)$, which follows from households' budget constraints.

At $s = T + t - 1$, from the first order condition of households, we have

$$\begin{aligned}
P^{\mathrm{A}}_{t+T-1} &= \beta C^{\mathrm{A}}_{t+T-1} \mathbb{E}\left[\frac{\theta_{t+T} + P^{\mathrm{A}}_{t+T}}{C^{\mathrm{A}}_{t+T}}\right], \\
P^{\mathrm{B}}_{t+T-1} &= \beta C^{\mathrm{B}}_{t+T-1} \mathbb{E}\left[\frac{\theta_{t+T} + P^{\mathrm{B}}_{t+T}}{C^{\mathrm{B}}_{t+T}}\right].
\end{aligned}$$

Substituting C^{A}_{t+T-1} and C^{B}_{t+T-1} with the households' budget constraints at $t + T - 1$, it follows that

$$\begin{aligned}
p^{\mathrm{A}}_{t+T-1}(\varepsilon) &= -p^{\mathrm{B}}_{t+T-1}(-\varepsilon), \\
c^{\mathrm{A}}_{t+T-1}(\varepsilon) &= c^{\mathrm{B}}_{t+T-1}(-\varepsilon).
\end{aligned}$$

Likewise, one can derive

$$\begin{aligned}
p^{\mathrm{A}}_{t}(\varepsilon) &= -p^{\mathrm{B}}_{t}(-\varepsilon), \\
c^{\mathrm{A}}_{t}(\varepsilon) &= c^{\mathrm{B}}_{t}(-\varepsilon).
\end{aligned}$$

On the other hand, one can rewrite $p_t^i(\varepsilon)$ as

$$\begin{aligned} p_t^{\mathrm{A}}(\varepsilon) &= c_t^{\mathrm{A}}(\varepsilon)\left(\sum_{j=1}^{T}\beta^j\mathbb{E}\left[\frac{\theta_{t+j}}{C_{t+j}^{\mathrm{A}}}\right]+\beta^T\mathbb{E}\left[\frac{P_{t+T}^{\mathrm{A}}}{C_{t+T}^{\mathrm{A}}}\right]\right) \\ &= -p_t^{\mathrm{B}}(-\varepsilon) \\ &= c_t^{\mathrm{B}}(-\varepsilon)\left(\sum_{j=1}^{T}\beta^j\mathbb{E}\left[\frac{\theta_{t+j}}{C_{t+j}^{\mathrm{B}}}\right]+\beta^T\mathbb{E}\left[\frac{P_{t+T}^{\mathrm{B}}}{C_{t+T}^{\mathrm{B}}}\right]\right). \end{aligned}$$

When $T\to\infty$, according to the TVC in market A and B,

$$\begin{aligned} \lim_{T\to\infty} -\beta^T\frac{P_T^{\mathrm{A}}}{C_T^{\mathrm{A}}}y_T^{\mathrm{A}} &= 0, \\ \lim_{T\to\infty} \beta^T\frac{P_T^{\mathrm{B}}}{C_T^{\mathrm{B}}}y_T^{\mathrm{B}} &= 0. \end{aligned}$$

If the steady state prices $\lim_{T\to\infty} P_{t+T}^i \neq 0$, then it must hold $y_{t+T}^i \neq 0$ in equilibrium. Otherwise, some IM can make an arbitrage profit by increasing liquidity providing. Thus, in this case,

$$\lim_{T\to\infty}\beta^T\mathbb{E}\left[\frac{P_{t+T}^{\mathrm{A}}}{C_{t+T}^{\mathrm{A}}}\right] = -\lim_{T\to\infty}\beta^T\mathbb{E}\left[\frac{P_{t+T}^{\mathrm{B}}}{C_{t+T}^{\mathrm{B}}}\right] = 0. \tag{II.18}$$

Else if $\lim_{T\to\infty} P_{t+T}^i = 0$, Equation (II.18) obviously holds as well.

Therefore, we have

$$\begin{aligned} p_t^{\mathrm{A}}(\varepsilon) &= c_t^{\mathrm{A}}(\varepsilon)\lim_{T\to\infty}\left(\sum_{j=1}^{T}\beta^j\mathbb{E}\left[\frac{\theta_{t+j}}{c_{t+j}^{\mathrm{A}}}\right]+\beta^T\mathbb{E}\left[\frac{p_{t+T}^{\mathrm{A}}}{c_{t+T}^{\mathrm{A}}}\right]\right) \\ &= c_t^{\mathrm{A}}(\varepsilon)\left(\sum_{j=1}^{\infty}\beta^j\mathbb{E}\left[\frac{\theta_{t+j}}{C_{t+j}^{\mathrm{A}}}\right]\right) \\ &= -p_t^{\mathrm{B}}(-\varepsilon) \\ &= c_t^{\mathrm{B}}(-\varepsilon)\left(\sum_{j=1}^{\infty}\beta^j\mathbb{E}\left[\frac{\theta_{t+j}}{C_{t+j}^{\mathrm{B}}}\right]\right). \end{aligned}$$

$\square$

II.A.2 Proof of Proposition II.1

Proof. From Lemma 1, we know that

$$p_t^{\mathrm{A}}(\varepsilon) = -p_t^{\mathrm{B}}(-\varepsilon).$$

Also, from households' budget constraints,

$$\begin{aligned} c_t^{\mathrm{A}}(\varepsilon) &= -p_t^{\mathrm{A}}(\varepsilon)\left(y_t^{\mathrm{A}} - y_{t-1}^{\mathrm{A}}\right) + w\left(K_{t-1}\right) + \left(u + y_{t-1}^{\mathrm{A}}\right)\varepsilon \\ &= p_t^{\mathrm{A}}(\varepsilon)\left(x_t - x_{t-1}\right)/l + w\left(K_{t-1}\right) + \left(u - x_{t-1}/l\right)\varepsilon, \end{aligned}$$

where $w(K) = a\gamma K^{\alpha} L^{\gamma} + b$, is a function of K.

Similarly,

$$c_t^{\mathrm{B}}(\varepsilon) = -p_t^{\mathrm{B}}(\varepsilon)\left(x_t - x_{t-1}\right)/l + w\left(K_{t-1}\right) - \left(u - x_{t-1}/l\right)\varepsilon.$$

Thus, it is obvious that

$$\begin{aligned} c_t^{\mathrm{A}}(\varepsilon) &= c_t^{\mathrm{B}}(-\varepsilon), \\ \mathbb{E}\left[\frac{\theta_{t+s} + P_{t+s}^{\mathrm{A}}}{C_{t+s}^{\mathrm{A}}}\right] &= -\mathbb{E}\left[\frac{\theta_{t+s} + P_{t+s}^{\mathrm{B}}}{C_{t+s}^{\mathrm{B}}}\right], \qquad \forall s \in \{1, 2, \dots\}. \end{aligned}$$

From households' first order conditions, it follows that

$$\begin{aligned} \phi_t &\equiv P_t^{\mathrm{B}} - P_t^{\mathrm{A}} \\ &= \beta C_t^{\mathrm{B}} \mathbb{E}\left[\frac{\theta_{t+1} + P_{t+1}^{\mathrm{B}}}{C_{t+1}^{\mathrm{B}}}\right] - \beta C_t^{\mathrm{A}} \mathbb{E}\left[\frac{\theta_{t+1} + P_{t+1}^{\mathrm{A}}}{C_{t+1}^{\mathrm{A}}}\right] \\ &= \beta\left(C_t^{\mathrm{A}} + C_t^{\mathrm{B}}\right) \mathbb{E}\left[\frac{\theta_{t+1} + P_{t+1}^{\mathrm{B}}}{C_{t+1}^{\mathrm{B}}}\right]. \end{aligned}$$

Thus,

$$P_t^{\mathrm{B}} = \beta C_t^{\mathrm{B}} \mathbb{E}\left[\frac{\theta_{t+1} + P_{t+1}^{\mathrm{B}}}{C_{t+1}^{\mathrm{B}}}\right] = \frac{C_t^{\mathrm{B}}}{C_t^{\mathrm{A}} + C_t^{\mathrm{B}}} \phi_t.$$

Likewise,

$$P_t^{\mathrm{A}} = -\frac{C_t^{\mathrm{A}}}{C_t^{\mathrm{A}} + C_t^{\mathrm{B}}}\phi_t.$$

Substituting C_t^i with HH's budget constraints in i, $i \in \{\mathrm{A}, \mathrm{B}\}$, one can obtain the following after rearranging

$$\begin{aligned} P_t^{\mathrm{B}} &= \frac{w\left(K_{t-1}\right) - \left(u - x_{t-1}/l\right)\theta_t}{2w\left(K_{t-1}\right)}\phi_t, \\ P_t^{\mathrm{A}} &= -\frac{w\left(K_{t-1}\right) + \left(u - x_{t-1}/l\right)\theta_t}{2w\left(K_{t-1}\right)}\phi_t. \end{aligned}$$

On the other hand, if we continue decompose ϕ_t,

$$\begin{aligned} \phi_t &= \beta C_t^{\mathrm{B}}\mathbb{E}\left[\frac{\theta_{t+1} + P_{t+1}^{\mathrm{B}}}{C_{t+1}^{\mathrm{B}}}\right] - \beta C_t^{\mathrm{A}}\mathbb{E}\left[\frac{\theta_{t+1} + P_{t+1}^{\mathrm{A}}}{C_{t+1}^{\mathrm{A}}}\right] \\ &= \beta\left[-P_t^{\mathrm{B}}\left(x_t - x_{t-1}\right)/l + w\left(K_{t-1}\right) - \left(u - x_{t-1}/l\right)\theta_t\right]\mathbb{E}\left[\frac{\theta_{t+1} + P_{t+1}^{\mathrm{B}}}{C_{t+1}^{\mathrm{B}}}\right] \\ &\quad - \beta\left[P_t^{\mathrm{A}}\left(x_t - x_{t-1}\right)/l + w\left(K_{t-1}\right) + \left(u - x_{t-1}/l\right)\theta_t\right]\mathbb{E}\left[\frac{\theta_{t+1} + P_{t+1}^{\mathrm{A}}}{C_{t+1}^{\mathrm{A}}}\right] \\ &= \beta\left[-\phi_t\left(x_t - x_{t-1}\right)/l + 2w\left(K_{t-1}\right)\right]\mathbb{E}\left[\frac{\theta_{t+1} + P_{t+1}^{\mathrm{B}}}{C_{t+1}^{\mathrm{B}}}\right]. \end{aligned}$$

After rearranging and repeatedly substituting with HH's first order condition, we can get

$$\begin{aligned} \phi_t &\equiv P_t^{\mathrm{B}} - P_t^{\mathrm{A}} \\ &= \frac{2w\left(K_{t-1}\right)}{M_t + \left(x_t - x_{t-1}\right)/l}. \end{aligned}$$

where

$$1/M_t := \sum_{j=1}^{\infty}\beta^j\mathbb{E}_t\left[\frac{\theta_{t+j}}{C_{t+j}^{\mathrm{B}}}\right],$$

which is independent of the realization of θ_t.

As IM take a net zero position in the financial markets, when u_t is constant, their optimization problems are deterministic. Accordingly, x_t, x_{t-1}, K_{t-1} are all independent of the realization of θ_t in equilibrium. Hence the price difference ϕ_t doesn't depend on any particular θ_t realization.

□

II.A.3 Proof of Proposition II.2

Proof. First, we prove for $\rho > \bar{\rho}$, any $x_t \neq ul$ or $\phi_t \neq 0$ won't hold in equilibrium. Second, we prove for $\rho \leq \bar{\rho}$, when the collateral constraint is slack for IM, it must hold that $x_t \leq 0$ and $\phi_t > 0$. Last, we prove for $\rho \leq \bar{\rho}$, when the collateral constraint is binding for IM, $x_t \in (0, ul)$. The pricing for the price gap follows naturally from the collateral constraint.

As proved in Proposition II.5, when $\rho > \bar{\rho}$, the steady state price gap and market liquidity are $\phi^* = 0$ and $x^* = ul$. Suppose IM only reaches the steady state position $x^* = ul$ in period t and before t they choose $x_s \neq ul$ in equilibrium, $\forall s < t$. Without loss of generality, we suppose in $t-1$, IM choose $x_{t-1} \neq ul$ and from the pricing formula in equilibrium $\phi_{t-1} \neq 0$. We know that in period t, $\phi_t = 0$. Thus, in $t-1$, IM are not collateral constrained. Given $\phi_{t-1} \neq 0$, a certain IM can make an arbitrage profit by taking $x'_{t-1} = x_{t-1} + \Delta x$ such that $\phi_{t-1}\Delta x > 0$ without assuming any increased obligation in t. Therefore, $x_{t-1} \neq ul$ cannot hold in equilibrium. Thus, $x_{t-1} = ul$ and $\phi_{t-1} = 0$ must hold. Similarly, this applies to all $s < t$.

As proved in Proposition II.5, when $\rho \leq \bar{\rho}$, the steady state price gap and market liquidity are $\phi^* > 0$ and $0 < x^* < ul$. Suppose in this case when the collateral constraint is slack for IM, $x_t > 0$ holds in equilibrium. Then from the budget constraints and the first order condition of IM, we have

$$\begin{aligned} c_t^{\mathrm{IM}} &= F\left(K_{t-1}\right) + (1-\delta)K_{t-1} - \left(x_{t-1} - x_t\right)\phi_t - K_t, \\ c_{t+1}^{\mathrm{IM}} &= F\left(K_t\right) + (1-\delta)K_t - \left(x_t - x_{t+1}\right)\phi_{t+1} - K_{t+1}, \end{aligned}$$

$$F'\left(K_t\right) + 1 - \delta = \frac{\phi_{t+1}}{\phi_t}, \tag{II.19}$$

and

$$-x_t\phi_{t+1} + (1-\delta)K_t > 0.$$

Assume certain IM choose to take $x'_t = x_t - \epsilon$ and $K'_t = K_t - \epsilon\phi_t$ in t, where $\epsilon\phi_t > 0$ and $-x'_t\phi_{t+1} + (1-\delta)K'_t > 0$. From the budget constraint, we can find that in this way its consumption in t stay the same as others, i.e. $c'_t = c_t^{\mathrm{IM}}$. However, in $t+1$ if this IM choose the same K_{t+1} and x_{t+1}, then

$$\begin{aligned} c'_{t+1} &= F\left(K'_t\right) + (1-\delta)K'_t - \left(x'_t - x_{t+1}\right)\phi_{t+1} - K_{t+1} \\ &= F\left(K_t - \epsilon\phi_t\right) + (1-\delta)\left(K_t - \epsilon\phi_t\right) - \left(x_t - \epsilon - x_{t+1}\right)\phi_{t+1} - K_{t+1}, \\ c'_{t+1} - c^{\mathrm{IM}}_{t+1} &= F\left(K_t - \epsilon\phi_t\right) - F\left(K_t\right) + (1-\delta)\left(-\epsilon\phi_t\right) + \epsilon\phi_{t+1} \\ &= F\left(K_t - \epsilon\phi_t\right) - F\left(K_t\right) + F'\left(K_t\right)\epsilon\phi_t > 0. \end{aligned}$$

The last equation follows from Equation (II.19), and the inequality holds because $F(\cdot)$ is a concave function of K_t. Thus, these IM can increase their utility by deviating from x_t. Therefore, if $\rho \leq \bar{\rho}$, $x_t > 0$ does not hold in equilibrium under the slack collateral constraints. The relationship between price spreads between t and $t+1$ follows from Equation (II.19).

Next we prove when the collateral constraints are binding for IM in t, then their positions in equilibrium must satisfy $x_t \in (0, ul)$. In particular, we show this by invalidating the cases $x_t \leq 0$ and $x_t \geq ul$.

Suppose in equilibrium there exists $x_t \leq 0$ when $\rho \leq \bar{\rho}$ and IM's collateral constraints are binding in t, i.e. $x_t\phi_{t+1} = (1-\delta)K_t$. Thus, $\phi_{t+1} < 0$. From IM's first order condition with respect to x_t, it must also hold that $\phi_t < 0$. Otherwise, certain IM can make arbitrage profit by taking $x'_t > 0$. From HH's side, this means $P^{\mathrm{B}}_{t+1} < 0$ and $P^{\mathrm{B}}_t < 0$, as $P^{\mathrm{B}}_t = C^{\mathrm{B}}_t\phi_t / \left(C^{\mathrm{A}}_t + C^{\mathrm{B}}_t\right)$.

Since $x_t < 0$, it follows

$$\begin{aligned} \mathbb{E}\left[\frac{\theta_{t+1}}{C^{\mathrm{B}}_{t+1}}\right] &= \mathbb{E}\left[\frac{\theta_{t+1}}{w\left(K_t\right) + P^{\mathrm{B}}_{t+1}\left(x_t - x_{t+1}\right)/l - \left(u - x_t/l\right)\theta_{t+1}}\right] \\ &= \mathbb{E}\left[\frac{\theta_{t+1}}{\left(w\left(K_t\right) - \left(u - x_t/l\right)\theta_{t+1}\right)\left(1 + \frac{\phi_{t+1}\left(x_t - x_{t+1}\right)/l}{2w\left(K_t\right)}\right)}\right] > 0. \end{aligned}$$

The last equation comes from

$$P^{\mathrm{B}}_{t+1} = \frac{w\left(K_t\right) - \left(u - x_t/l\right)\theta_{t+1}}{2w\left(K_t\right)}\phi_{t+1}.$$

The inequality holds because when $\theta_{t+1} = \epsilon > 0$, $c^{\mathrm{B}}_{t+1}(\epsilon) < c^{\mathrm{B}}_{t+1}(-\epsilon)$ when $x_t \leq 0$.

Assume that from $t+1$ onwards, IM's position sequence $\{x_{t+s}\}$ in equilibrium stays below ul, i.e. $x_{t+s} < ul$, until some period $t+T$, for $s \in \{1, 2, \dots, T-1\}$. Thus,

$$\mathbb{E}\left[\frac{\theta_{t+s}}{C^{\mathrm{B}}_{t+s}}\right] > 0, \qquad \forall s \leq T.$$

If $T = \infty$, then from the pricing formula,

$$\frac{P^{\mathrm{B}}_t}{C^{\mathrm{B}}_t} = \lim_{T\to\infty} \beta \mathbb{E}_t\left[\frac{\theta_{t+1}}{C^{\mathrm{B}}_{t+1}}\right] + \ldots + \beta^T \mathbb{E}_t\left[\frac{\theta_{t+T}}{C^{\mathrm{B}}_{t+1}}\right] + \beta^T \mathbb{E}_t\left[\frac{P^{\mathrm{B}}_{t+T}}{C^{\mathrm{B}}_{t+T}}\right],$$

one can conclude that if $P^{\mathrm{B}}_t < 0$,

$$\lim_{T\to\infty} \beta^T \mathbb{E}_t\left[\frac{P^{\mathrm{B}}_{t+T}}{C^{\mathrm{B}}_{t+T}}\right] < 0$$

must hold. However, this violates HH's TVC.

Else if $T < \infty$, i.e. $x_{t+T} \geq ul$, suppose $P^{\mathrm{B}}_{t+T} \leq 0$ holds. In this case if $\phi_{t+T+1} \leq 0$, then at $t+T$ IM's collateral constraints are not binding, $x_{t+T}\phi_{t+T+1} \leq 0$. However, from previous proof, we know that if they slack, it must have $x_{t+T} < 0$. Thus, $P^{\mathrm{B}}_{t+T} \leq 0$ cannot hold. Now suppose $\phi_{t+T+1} > 0$. However, given $\phi_{t+T} < 0$ and $\phi_{t+T+1} > 0$, certain competitive IM can make arbitrage profit by changing their positions from $x_{t+T} \geq ul$ to $x'_{t+T} < 0$.

Therefore, when $x_t \leq 0$, $\phi_t \leq 0$ and $\phi_{t+1} \leq 0$ cannot possibly hold. Thus, if $x_t \leq 0$, we must have $\phi_t > 0$ and $\phi_{t+1} > 0$. However, this contradicts with the binding condition, i.e. $x_t\phi_{t+1} = (1-\delta)K_t > 0$. Thus, $x_t \leq 0$ cannot hold in equilibrium when the collateral constraints are binding. In the same fashion, when $x_t \geq ul$, we can derive $\phi_{t+1} \leq 0$. This again doesn't satisfy IM's binding conditions. Consequently, when IM are collateral constrained in equilibrium, $x_t \in (0, ul)$. By definition, $\phi_{t+1} = (1-\delta)K_t/x_t$.

□

II.A.4 Proof of Proposition II.3

Proof. We start proving the proposition by assuming that some IM only living for $T = 2$ periods. We suppose further that all IM's collateral constraints are binding in $t = 1$ and their initial wealth is W_1. There the two-period living IM's optimization problems become

$$\max_{C_1^{\mathrm{IM}}, C_2^{\mathrm{IM}}, x_1, K_1} \log\left(C_1^{\mathrm{IM}}\right) + \rho \log\left(C_2^{\mathrm{IM}}\right),$$

subject to

$$\begin{aligned}
(i) \qquad & C_1^{\mathrm{IM}} + K_1 = W_1 + x_1 \phi_1, \\
(ii) \qquad & C_2^{\mathrm{IM}} = F\left(K_1\right) + (1-\delta) K_1 - x_1 \phi_2, \\
(iii) \qquad & (1-\delta) K_0 \geq x_0 \phi_1.
\end{aligned}$$

As they have binding collateral constraints, $(1-\delta)K_1 = x_1\phi_2$.

Applying first order conditions, we have

$$\begin{aligned}
& -\frac{1}{C_1^{\mathrm{IM}}} + \frac{\rho\left(F'\left(K_1\right) + 1 - \delta\right)}{C_2^{\mathrm{IM}}} + \lambda_1(1-\delta) = 0, \\
& \frac{\phi_1}{C_1^{\mathrm{IM}}} - \frac{\phi_2}{C_2^{\mathrm{IM}}} - \lambda_1 \phi_2 = 0,
\end{aligned}$$

where $\lambda_1 > 0$ is the Lagrange multiplier for the collateral constraint at $t = 1$.

Solving all the above equations, we obtain

$$\begin{aligned}
C_1^{\mathrm{IM}} &= \frac{W_1}{1+\alpha\rho}, \\
K_1 &= \frac{\alpha\rho W_1}{(1+\alpha\rho) S_1}, \qquad \text{where } S_1 = 1 - \frac{(1-\delta)\phi_1}{\phi_2} = \frac{1}{\mu_1}.
\end{aligned}$$

Similarly, for some IM living for periods of $T \in \{3, \ldots\}$, and IM's collateral constraints are binding for $t \in \{1, 2, \ldots, T-1\}$, we obtain

$$\begin{aligned}
C_1^{\mathrm{IM}} &= \frac{W_1}{1+\alpha\rho+\alpha^2\rho^2+\ldots+\alpha^{T-1}\rho^{T-1}}, \\
K_1 &= \frac{\left(\alpha\rho+\alpha^2\rho^2+\ldots+\alpha^{T-1}\rho^{T-1}\right)\mu_1 W_1}{1+\alpha\rho+\alpha^2\rho^2+\ldots+\alpha^{T-1}\rho^{T-1}}.
\end{aligned}$$

When $T \to \infty$,

$$\begin{aligned} C_1^{\mathrm{IM}} &= \lim_{T \to \infty} \frac{W_1}{1 + \alpha\rho + \alpha^2\rho^2 + \cdots + \alpha^{T-1}\rho^{T-1}} = (1 - \alpha\rho)W_1. \\ K_1 &= \alpha\rho\mu_1 W_1. \end{aligned}$$

Likewise, starting with period t and when

$$W_t = F\left(K_{t-1}\right) + (1 - \delta)K_{t-1} - x_{t-1}\phi_t = F\left(K_{t-1}\right)$$

due to binding collateral constraints, obviously we can extend the above to period t, for all $t = 1, 2, \ldots$.

$$\begin{aligned} C_t^{\mathrm{IM}} &= (1 - \alpha\rho)W_t. \\ K_t &= \alpha\rho\mu_t W_t. \\ W_{t+1} &= F\left(K_t\right). \end{aligned}$$

Also one can easily check that with binding collateral constraints, the steady state level of IM's consumption, capital investment and market liquidity, as shown in Proposition 5, are also consistent with IM's TVC.

□

II.A.5 Proof of Proposition II.4

Proof. From Proposition II.2, we know that for given technology and endowment shocks, if $\rho > \bar{\rho}$, the economy resembles one in the neoclassical model with frictionless financial markets, i.e. $x_t = ul$. Obviously, in this instance, the equilibrium exists.

Otherwise, if $\rho \leq \bar{\rho}$, when IM's collateral constraints are binding, one can solve the equilibrium backwards through Equation (II.12), (II.13), (II.15), (II.16), (II.17). When the collateral constraints are slack for the initial few periods, one can also solve the equilibrium through Equation (II.14), (II.4), (II.12), as well as both types of agents' budget constraints. Meanwhile, IM hold opposite positions in two markets and $x_t^i = -y_t^i l$ ensures that markets clear for financial assets.

With the assumption of the shock intensity being constant, i.e. $u_t = u$, and IM having net zero positions of financial assets, IM are not exposed to any idiosyncratic shocks from θ_t, IM's optimization problems are deterministic ones. Hence, in equilibrium the quantities $\left(\phi_t,\ x_t,\ y_t^i,\ K_t\right)$ are also deterministic.

As implied by Proposition II.2, in equilibrium IM's position in the markets with positive shock intensity will not exceed the total asset demands, i.e. $x_t \leq ul$. Therefore, $\phi_t \geq 0$ and from Equation (II.10) and (II.11), $P_t^{\mathrm{A}} \leq 0$ and $P_t^{\mathrm{B}} \geq 0$.

□

II.A.6 Proof of Proposition II.5

Proof. To facilitate later proving, first we derive the steady state price spreads in dependent of collateral constraints being slack or not.

$$
\begin{aligned}
\phi_t^* &:= P_t^{\mathrm{B}*} - P_t^{\mathrm{A}*} \\
&= \beta C_t^{\mathrm{B}*} \mathbb{E}\left[\frac{P_{t+1}^{\mathrm{B}*} + \theta_{t+1}}{C_{t+1}^{\mathrm{B}*}}\right] - \beta C_t^{\mathrm{A}*} \mathbb{E}\left[\frac{P_{t+1}^{\mathrm{A}*} + \theta_{t+1}}{C_{t+1}^{\mathrm{A}*}}\right] \\
&= \beta \left(C_t^{\mathrm{B}*} + C_t^{\mathrm{A}*}\right) \mathbb{E}\left[\frac{P_{t+1}^{\mathrm{B}*} + \theta_{t+1}}{C_{t+1}^{\mathrm{B}*}}\right] \\
&= 2\beta w\left(K^*\right) \left(\frac{\phi_{t+1}}{C_{t+1}^{\mathrm{A}*} + C_{t+1}^{\mathrm{B}*}} + \mathbb{E}\left[\frac{\theta_{t+1}}{C_{t+1}^{\mathrm{B}*}}\right]\right) \\
&= 2\beta w\left(K^*\right) \left(\frac{\phi_{t+1}^*}{2w\left(K^*\right)} + \mathbb{E}\left[\frac{\theta_{t+1}}{w\left(K^*\right) - \left(u - x^*/l\right)\theta_{t+1}}\right]\right),
\end{aligned}
$$

where the first equation is by definition, the second is derived directly from the first order condition of HH, the third one is from Proposition 1 and Lemma 1, the fourth equation follows from $P_t^{\mathrm{B}*} = \frac{\phi_t C_t^{\mathrm{B}}}{C_t^{\mathrm{A}} + C_t^{\mathrm{B}}}$, and the final equation is straight derived by HH's steady state budget constraints. Thus, rearranging the above and equating $\phi_t^* = \phi_{t+1}^*$ as the steady state property, we get

$$
\phi^* = \frac{2\beta w\left(K^*\right)}{1-\beta} \mathbb{E}\left[\frac{\theta}{w\left(K^*\right) - \left(u - x^*/l\right)\theta}\right].
$$

It follows straightforward that when $x^* = ul$, $\phi^* = 0$, and when $x^* < ul$, $\phi^* > 0$.

Part 1 - Proof of the Slack Steady State

Given the steady state is one with slack collateral constraints, as IM are competitive, it must satisfy $x^* = ul$ and $\phi^* = 0$. Suppose it doesn't, e.g. $\phi^* \neq 0$ or $x^* \neq ul$. If $\phi^* \neq 0$, since IM are not collateral constrained, they will increase or decrease their position to arbitrage the price difference, until in equilibrium $\phi^* = 0$. Likewise if $x^* < ul$ (or $x^* > ul$), from the HH's pricing we must have $\phi^* > 0$ ($\phi^* < 0$). Again, IM will make sure $x^* = ul$ and $\phi^* = 0$.

Meanwhile, as in the slack steady state all arbitrage opportunities are eliminated. Thus the IM's budget constraints in the long run is equivalent to

$$C_t^{\mathrm{IM}} = F\left(K_{t-1}\right) + (1-\delta)K_{t-1} - K_t.$$

Combining with the first order condition and equating $K_t = K_{t-1} = \ldots = K^*$, one can yield $K^* = K_s^* = F'^{-1}\left(\frac{1-\rho(1-\delta)}{\rho}\right)$, which is the same as that in neoclassical growth model.

Part 2 - Proof of the Binding Steady States

As the collateral constraints are binding in these states, we can have $(1-\delta)K^* = x^*\phi^*$. From the first order conditions of IM, by equating capital investment, asset positions and spreads across periods to $(K^*,\ x^*,\ \phi^*)$ we can derive $K_b^* = F'^{-1}(\delta/\rho)$. Since $\delta/\rho < \left(1-\rho(1-\delta)\right)/\rho$ and $F'^{-1}(K)$ is a decreasing function of K, one can conclude that $K_b^* > K_s^*$.

Part 3 - Proof of the Existence of the Cutoff Value

Define a function $k(\rho) := F'^{-1}(\delta/\rho)$ as the potential long-run capital investment level, given that the equilibrium can support a steady state with binding collateral constraints and when IM's discount factor is ρ.

Also we define the following function of IM's discount factor ρ and steady state position x^* as the corresponding steady state price spread:

$$\Phi(x^*,\rho) := \frac{2\beta}{1-\beta} w(k(\rho)) \mathbb{E}\left[\frac{\theta}{w(k(\rho)) - \left(u - x^*/l\right)\theta}\right].$$

Since if there is a binding steady state, it must satisfy the collateral constraint $(1-\delta)k(\rho) = x^*\Phi(x^*,\rho)$. Thus we construct a product function of steady state position and spread $G(x,\rho)$ and an auxiliary function $g(x,\rho,\theta)$.

$$\begin{aligned} G(x,\rho) &:= x\Phi(x,\rho) = x\frac{2\beta}{1-\beta}w(k(\rho))\mathbb{E}\left[\frac{\theta}{w(k(\rho))-(u-x/l)\theta}\right]. \\ g(x,\rho,\theta) &:= \frac{x\theta}{w(k(\rho))-(u-x/l)\theta}. \end{aligned}$$

Because

$$\frac{\partial g(x,\rho,\theta)}{\partial x} = \frac{\theta w(k(\rho)) - u\theta^2}{\left(w(k(\rho))-(u-x/l)\theta\right)^2}$$

and

$$\frac{\partial g^2(x,\rho,\theta)}{\partial x^2} = \frac{-2\theta^2\left(w(k(\rho))-u\theta\right)}{l\left(w(k(\rho))-(u-x/l)\theta\right)^3} < 0,$$

where $x \in [0, ul]$, $g(x,\rho,\theta)$ is a strict local concave function of x over $[0, ul]$, given ρ and θ. Note that here we apply $w(k(\rho)) - u\theta > e_t^i - u\bar{\theta} > 0$.

Similarly, since

$$G(x,\rho) = \frac{2\beta}{1-\beta}w(k(\rho))\int_{-\bar{\theta}}^{\bar{\theta}} g(x,\rho,\theta)p(\theta)\mathrm{d}\theta,.$$

where $p(\theta)$ is the PDF of θ, $G(x,\rho)$ is also a strict local concave function with respect to x over $[0, ul]$ given ρ. Moreover,

$$\begin{aligned} \left.\frac{\partial G(x,\rho)}{\partial x}\right|_{x=0} &= \left.\int_{-\bar{\theta}}^{\bar{\theta}} \frac{\theta w(k(\rho)) - u\theta^2}{\left(w(k(\rho))-(u-x/l)\theta\right)^2}p(\theta)\mathrm{d}\theta\right|_{x=0} \\ &= \int_{-\bar{\theta}}^{\bar{\theta}} \frac{\theta}{\left(w(k(\rho))-u\theta\right)}p(\theta)\mathrm{d}\theta > 0, \end{aligned}$$

$$\begin{aligned} \left.\frac{\partial G(x,\rho)}{\partial x}\right|_{x=ul} &= \left.\int_{-\bar{\theta}}^{\bar{\theta}} \frac{\theta w(k(\rho)) - u\theta^2}{\left(w(k(\rho))-(u-x/l)\theta\right)^2}p(\theta)\mathrm{d}\theta\right|_{x=ul} \\ &= \int_{-\bar{\theta}}^{\bar{\theta}} \frac{\theta w(k(\rho)) - u\theta^2}{\left(w(k(\rho))\right)^2}p(\theta)\mathrm{d}\theta \\ &= \int_{-\bar{\theta}}^{\bar{\theta}} \frac{-u\theta^2}{\left(w(k(\rho))\right)^2}p(\theta)d\theta < 0, \end{aligned}$$

Therefore, given ρ, $G(x,\rho)$ has a unique interior maximal value on some $\bar{x} \in (0, ul)$, where

$$\left.\frac{\partial G(x,\rho)}{\partial x}\right|_{x=\bar{x}} = 0.$$

On the other hand, rearranging $G(x,\rho)$,

$$\begin{aligned} G(x,\rho) &= \frac{2\beta}{1-\beta}\int_{-\bar{\theta}}^{\bar{\theta}} \frac{w(k(\rho))x\theta}{w(k(\rho))-(u-x/l)\theta} p(\theta)\mathrm{d}\theta \\ &= \frac{2\beta}{1-\beta}\int_{-\bar{\theta}}^{\bar{\theta}} \frac{x\theta}{1-(u-x/l)\theta/w(k(\rho))} p(\theta)\mathrm{d}\theta. \end{aligned}$$

It is straightforward to show $\partial G(x,\rho)/\partial\rho < 0$. Thus, $G(x,\rho)$ is a decreasing function of ρ.

To prove for the existence of a cut-off $\bar{\rho}$, which determines IM's discount factor ρ corresponding to a slack steady state or a binding one, we define new auxiliary functions $H(\rho)$ and $Q(\rho)$:

$$\begin{aligned} H(\rho) &:= \max_x G(x,\rho), \\ Q(\rho) &:= H(\rho) - (1-\delta)k(\rho). \end{aligned}$$

If the economy supports binding steady states, it must satisfy $Q(\rho) \geq 0$. When $Q(\rho) < 0$, it means that the collateral is sufficient to eliminate any arbitrage opportunities and can only support steady states with slack collateral constraints. This is because that if there exist binding steady states, then given ρ, $G(x^*,\rho) - (1-\delta)k(\rho) = 0$ should have real solutions of $x^* \in (0, ul)$. Since $G(x^*,\rho) - (1-\delta)k(\rho) < Q(\rho)$, if $Q(\rho) < 0$, then there is no solution of x^* given this value of ρ that can support any binding steady state. As $Q(\rho)$ is a continuous and strictly decreasing function of ρ, there exists a unique cutoff value $\bar{\rho}$, such that $Q(\bar{\rho})) = 0$. That is, if ρ above this $\bar{\rho}$, $Q(\rho) < 0$, there is only slack steady state. Whereas when $\rho \leq \bar{\rho}$, $G(x^*,\rho) - (1-\delta)k(\rho) = 0$ has real solutions on $(0, ul)$, there exist binding steady states.

$\square$

II.A.7 Proof of Proposition II.6

Proof. To prove for the existence of two distinct steady states, for any given $\rho' \in (0, \bar{\rho}]$, define an auxiliary function $J(x, \rho') := G(x, \rho') - (1-\delta)k(\rho') \leq Q(\rho')$, where $G(x, \rho)$, $Q(\rho)$ and $k(\rho)$ is constructed in the proof of Proposition II.5. Thus, $Q(\rho') \geq 0$, for $\rho' \in [0, \bar{\rho}]$. Obviously, $J(x, \rho')$ is a continuous function and only when $x = \bar{x}$, which is the global maximizer, is $J(x, \rho')$ equal to $Q(\rho')$.

To prove for the existence of two distinct steady states, we adopt the functions $G(x, \rho)$, $Q(\rho)$ and $k(\rho)$ defined in Proposition II.5 and construct a new auxiliary function

$$J(x, \rho) := G(x, \rho) - (1-\delta)k(\rho) = x\Phi(x, \rho) - (1-\delta)k(\rho) \leq Q\left(\rho'\right).$$

Inheriting features from $G(x, \rho)$ and $k(\rho)$, it is obvious that $J(x, \rho)$ is a strictly concave function of x and a decreasing function of ρ. Moreover, when $J(x, \rho) = 0$ has real solutions in $(0, ul)$, it means that there exists binding steady states given ρ, and the solutions x^* correspond to the steady state market liquidity.

Also, by definition,

$$Q(\rho) = \max_x J(x, \rho).$$

From proof in Proposition II.5, we know that $Q(\rho)$ is a decreasing function of ρ and when $Q(\rho) < 0$, there is no binding steady states possible. When $Q(\rho) = 0$, $\rho = \bar{\rho}$. Meanwhile, when $\rho = \bar{\rho}$, if there exists any binding steady state at all, it must satisfy

$$J(x, \bar{\rho}) = 0.$$

Thus, $J(x, \bar{\rho}) = Q(\bar{\rho})$. Given $J(x, \rho)$ is a strictly concave function of x and given $J(x, \rho) = Q(\rho)$ only holds when x equals to the global maximizer $\bar{x}(\bar{\rho})$, we can conclude that $x^* = \bar{x}(\bar{\rho})$ is the only solution. Therefore, when $\rho = \bar{\rho}$, there is a unique binding steady state with $x^* = \bar{x}(\bar{\rho})$ as its market liquidity.

When $0 < \rho < \bar{\rho}$, then

$$Q(\rho) = \max_x J(x, \rho) > 0.$$

Denote $x(\rho)$ as the global maximizer of $J(x,\rho)$, given ρ. Since by definition,

$$J(0,\rho) < 0, \quad J(ul,\rho) < 0,$$

given $J(x,\rho)$ is a strictly concave function of x, there must exist two distinct solutions $x_1^* \in (0,\bar{x}(\rho))$ and $x_2^* \in (\bar{x}(\rho), ul)$ to $J(x,\rho) = 0$. Thus, when $0 < \rho < \bar{\rho}$, there exist two distinct binding steady states with different levels of market liquidity, i.e. $x_1^* < x_2^*$.

As both steady states satisfy the binding collateral constraints $x_j^* \phi_j^* = (1-\delta)K_b^*$, $j \in \{1,2\}$, where $K_b^* = k(\rho)$ is a strictly increasing function of ρ. Thus, the two steady states share the same capital investment level $K_b^* = k(\rho)$. On the other hand, if x^* in one is smaller than the other, the steady state price spread must be higher than the other. That is, if $x_1^* < x_2^*$, then $\phi_1^* > \phi_2^*$.

From HH's perspective, as in both steady states they have the same labor income and $w(k(\rho))$ because of the common K^*, higher x^* reduces their consumption volatility. Thus HH's utility is strictly higher in SS_2 with larger market liquidity $x_2^* > x_1^*$.

Furthermore, as $J(x,\rho)$ is a decreasing function of ρ. Thus, $J(x,\rho_1) < J(x,\rho_2)$, if $\rho_1 > \rho_2$. On the other hand,

$$\begin{aligned}
\left.\frac{\partial J(x,\rho)}{\partial x}\right|_{x\in(0,\bar{x}(\rho))} &= \left.\frac{\partial G(x,\rho)}{\partial x}\right|_{x\in(0,\bar{x}(\rho))} > 0, \\
\left.\frac{\partial J(x,\rho)}{\partial x}\right|_{x\in(\bar{x}(\rho),ul)} &= \left.\frac{\partial G(x,\rho)}{\partial x}\right|_{x\in(\bar{x}(\rho),ul)} < 0.
\end{aligned}$$

That is, for $x \in (0,\bar{x}(\rho))$, $J(x,\rho)$ is an increasing function of x. Thus, for $J(x_1,\rho_1) = J(x_1',\rho_2) = 0$, where $x_1,\ x_1' \in (0,\bar{x}(\rho))$, it must hold $x_1 > x_1'$. The market liquidity in SS_1 increases with ρ. Similarly, x_2^* in SS_2 decreases with ρ, i.e. $x_2' > x_2$. As $k(\rho)$ is an increasing function of ρ, $K_b^*(\rho_1) > K_b^*(\rho_2)$. From binding collateral constraints, $x_2\phi_2 = (1-\delta)K_b^*(\rho_1)$ and $x_2'\phi_2' = (1-\delta)K_b^*(\rho_2)$, we can conclude that $\phi_2 > \phi_2'$. Since it is easy to verify that

$$\frac{\partial \Phi(x,\rho)}{\partial x} < 0, \qquad \frac{\partial \Phi(x,\rho)}{\partial \rho} < 0,$$

we can also derive $\phi_1 < \phi_1'$.

□

II.A.8 Proof of Proposition II.7

Proof. If we modify the definition of functions $\Phi(x,\rho)$, $G(x,\rho)$, $g(x,\rho,\theta)$, $J(x,\rho)$, $H(\rho)$ and $Q(\rho)$ in the proof of Proposition II.5 and Proposition II.6 by extending them also as functions of the shock intensity u, then we get

$$\begin{aligned}
\Phi(x,\rho,u) &:= \phi^*\left(x^*,K^*\right) = \frac{2\beta}{1-\beta}w(k(\rho))\mathbb{E}\left[\frac{\theta}{w(k(\rho))-(u-x/l)\theta}\right],\\
G(x,\rho,u) &:= x\Phi(x,\rho,u) = x\frac{2\beta}{1-\beta}w(k(\rho))\mathbb{E}\left[\frac{\theta}{w(k(\rho))-(u-x/l)\theta}\right],\\
g(x,\rho,u,\theta) &:= \frac{x\theta}{w(k(\rho))-(u-x/l)\theta},\\
J(x,\rho,u) &:= G(x,\rho,u)-(1-\delta)k(\rho),\\
H(\rho,u) &:= \max_x G(x,\rho,u),\\
Q(\rho,u) &:= H(\rho,u)-(1-\delta)k(\rho).
\end{aligned}$$

In particular,

$$\frac{\partial g(x,\rho,u,\theta)}{\partial u} = \frac{x\theta^2}{\left(w(k(\rho))-(u-x/l)\theta\right)^2} \geq 0,$$

Similarly, $\partial G(x,\rho,u)/\partial u > 0$, $\partial J(x,\rho,u)/\partial u > 0$, for $x \in (0,ul]$. If follows naturally that $\partial H(\rho,u)/\partial u > 0$ and $\partial Q(\rho,u)/\partial u > 0$. Thus, suppose if $u_1 < u_2$, then $Q\left(\rho,u_1\right) < Q\left(\rho,u_2\right)$.

What determines the cutoff value $\bar{\rho}$ for a given u is the ρ' such that $Q\left(\rho',u\right)=0$. As $Q(\rho,u)$ is a decreasing function of ρ and an increasing function of u, to satisfy

$$\begin{aligned}
Q\left(\bar{\rho}_1,u_1\right) &= 0,\\
Q\left(\bar{\rho}_2,u_2\right) &= 0.
\end{aligned}$$

we must have $\bar{\rho}_1 < \bar{\rho}_2$ if $u_1 < u_2$. Thus, the cutoff value increases with u.

From the property of $G(x, \rho, u)$, it is apparent that $J(x, \rho, u)$ is a strictly concave function of x. By definition,

$$\left.\frac{\partial J(x,\rho,u)}{\partial x}\right|_{x=0} > 0;$$
$$\left.\frac{\partial J(x,\rho,u)}{\partial x}\right|_{x=ul} < 0;$$
$$\left.\frac{\partial J(x,\rho,u)}{\partial x}\right|_{x=\bar{x}} = 0.$$

$\partial J(x, \rho, u)/\partial x$ is a continuous function of x, thus, we can conclude $\partial J(x, \rho, u)/\partial x > 0$ for $x \in (0, \bar{x})$ and $\partial J(x, \rho, u)/\partial x < 0$ for $x \in (\bar{x}, ul)$. What determines the steady state x^* in both $x \in (0, \bar{x})$ and $x \in (\bar{x}, ul)$ is that they must satisfy $J(x^*, \rho, u) = 0$.

Previously we know $\partial J(x, \rho, u)/\partial u > 0$, so if $u_1 < u_2$, for a given x, $J(x, \rho, u_1) < J(x, \rho, u_2)$. As $J(x, \rho, u)$ is an increasing function of x at $x_1^*[u_j]$, $j \in \{1, 2\}$, to satisfy $J(x_1[u_1], \rho, u_1) = 0$ and $J(x_1[u_2], \rho, u_2) = 0$ simultaneously, we must have $x_1^*[u_1] > x_1^*[u_2]$. The similar also applies for SS_2, where the opposite holds, $x_2^*[u_1] < x_2^*[u_2]$.

Also, the steady state capital investment level $K_b^* = k(\rho)$ is only determined by ρ and stay independent of u. Thus, in the two economies with the same discount factor ρ of IM, $K_b^*[u_1] = K_b^*[u_2] = k(\rho)$.

□

II.A.9 Proof of Proposition II.8

Proof. To facilitate the proof, we first redefine

$$k(\rho, a) = F'^{-1}\left(\frac{\delta}{\rho}\right) = \left(\frac{\delta}{a\rho\alpha L^{\gamma}}\right)^{1/(\alpha-1)}$$

as the steady state capital investment level, given that the equilibrium can support a steady state with binding collateral constraints, IM's discount factor is ρ and the total productivity factor is a. It is straightforward that

$$\frac{\partial k(\rho, a)}{\partial a} > 0.$$

Thus, all else equal, $K_b^*[a_1] < K_b^*[a_2]$, if $a_1 < a_2$.

Also redefine the following function of steady state price difference in terms of ρ, a and steady state position x:

$$\Phi(x,\rho,a) := \frac{2\beta}{1-\beta} w(k(\rho,a))\mathbb{E}\left[\frac{\theta}{w(k(\rho,a)) - (u-x/l)\theta}\right].$$

Following the similar approach, we reconstruct the functions $G(x,\rho,a)$, $H(\rho,a)$ and $Q(\rho,a)$ by including the total productivity parameter a.

$$\begin{aligned}
G(x,\rho,a) &:= x\Phi(x,\rho,a) = x\frac{2\beta}{1-\beta} w(k(\rho,a))\mathbb{E}\left[\frac{\theta}{w(k(\rho,a)) - (u-x/l)\theta}\right], \\
H(\rho,a) &:= \max_x G(x,\rho,a), \\
Q(\rho,a) &:= H(\rho,a) - (1-\delta)k(\rho,a), \\
J(x,\rho,a) &:= G(x,\rho,a) - (1-\delta)k(\rho,a).
\end{aligned}$$

It is easy to show that

$$\begin{aligned}
\frac{\partial G(x,\rho,a)}{\partial a} &< 0, \qquad \frac{\partial G(x,\rho,a)}{\partial \rho} < 0; \\
\frac{\partial Q(\rho,a)}{\partial a} &< 0, \qquad \frac{\partial Q(\rho,a)}{\partial \rho} < 0.
\end{aligned}$$

The cutoff value $\bar{\rho}$ for IM's discount factor defined in Proposition II.5 is determined by $Q(\bar{\rho},a) = 0$. Suppose if $a_1 < a_2$, then $Q(\bar{\rho},a_1) > Q(\bar{\rho},a_2)$. The cutoff values must follow $\bar{\rho}_1 > \bar{\rho}_2$, so that

$$Q(\bar{\rho}_1,a_1) = 0 \quad \text{and} \quad Q(\bar{\rho}_2,a_2) = 0.$$

Thus, all else equal, the cutoff discount factor $\bar{\rho}$, which satisfy $Q(\bar{\rho},a) = 0$, decreases with a.

Given ρ, the steady state $x^*(a)$ is the solution to $J(x,\rho,a) = 0$. Since $J(x,\rho,a)$ is a strictly concave function with respect to x, and

$$\left.\frac{\partial J(x,\rho,a)}{\partial x}\right|_{x=0} > 0, \quad \left.\frac{\partial J(x,\rho,a)}{\partial \rho}\right|_{x=ul} < 0, \quad \left.\frac{\partial J(x,\rho,a)}{\partial \rho}\right|_{x=\bar{x}(a)} = 0,$$

where $\bar{x}(a) \in (0, ul)$ is the solution to $Q(x, \bar{\rho}, a) = 0$, $J(x, \rho, a)$ is an increasing function of x in $(0, \bar{x}(a))$ and a decreasing function in $(\bar{x}(a), ul)$. Also, for $\rho < \bar{\rho}$ by definition one can easily verify

$$J(0, \rho, a) < 0, \quad J(\bar{x}(a), \rho, a) > 0, \quad J(ul, \rho, a) < 0.$$

The second relation holds because $J(\bar{x}(a), \rho, a) = Q(\rho, a)$, $Q(\bar{\rho}, a) = 0$, and $\partial Q(\rho, a)/\partial \rho < 0$. Thus, there exist one unique $x_1^* \in (0, \bar{x}(a))$ and one unique $x_2^* \in (\bar{x}(a), ul)$ as solutions to $J(x^*, \rho, a) = 0$. In particular,

$$\frac{\partial J(x, \rho, a)}{\partial a} < 0.$$

If $a_1 < a_2$, to satisfy both $J(x_1, \rho, a_1) = 0$ and $J(x_1', \rho, a_2) = 0$, we must have $x_1 < x_1'$, where $x_1 \in (0, \bar{x}(a_1))$ and $x_1' \in (0, \bar{x}(a_2))$. Similarly, $x_2 > x_2'$ holds, where $x_2 \in (\bar{x}(a_1), ul)$ and $x_2' \in (\bar{x}(a_2), ul)$.

In the binding steady state, the collateral constraints hold with equality, i.e. $(1 - \delta)K^*[a_1] = x_2^*[a_1]\,\phi_2^*[a_1]$ and $(1 - \delta)K^*[a_2] = x_2^*[a_2]\,\phi_2^*[a_2]$. Since $K^*[a_1] < K^*[a_2]$ and $x_2^*[a_1] > x_2^*[a_2]$, it follows naturally $\phi_2^*[a_1] < \phi_2^*[a_2]$.

Also,

$$\frac{\partial \Phi(x, \rho, a)}{\partial x} < 0, \quad \frac{\partial \Phi(x, \rho, a)}{\partial a} < 0.$$

As $x_1^*[a_1] < x_1^*[a_2]$ and $a_1 < a_2$, it follows straightforward that $\phi_1^*[a_1] > \phi_1^*[a_2]$.

□

II.A.10 Proof of Corollary II.1

Proof. We first show the properties of capital investment K_t. First we claim that K_t increases with IM's wealth level. If it is a binding state, it follows from Proposition II.3, $\frac{\partial K_t}{\partial W_t} > 0$. Similarly, $\frac{\partial x_t}{\partial K_t} > 0$. Thus when IM's wealth reduces, e.g., $W_t < W^*$, both K_t and x_t also drop.

For the price spreads, given the previous steady state values x_{t-1}^* and K_{t-1}^*, from the pricing formulas derived from Proposition II.1 and Lemma 1, we have

$$\begin{aligned}
\phi_t &= \left(C_t^{\mathrm{A}} + C_t^{\mathrm{B}}\right) \left(\sum_{j=1}^{\infty} \beta^j \mathbb{E}\left[\frac{\theta_{t+j}}{C_{t+j}^{\mathrm{B}}}\right] \right) \\
&= \left(-\phi_t \left(x_t - x_{t-1}\right)/l + 2w\left(K_{t-1}\right)\right) \left(\sum_{j=1}^{\infty} \beta^j \mathbb{E}\left[\frac{\theta_{t+j}}{C_{t+j}^{\mathrm{B}}}\right] \right).
\end{aligned}$$

Rearraging the above yields

$$\begin{aligned}
\phi_t &= \frac{2w\left[K_{t-1}^*\right]}{\left(\sum_{j=1}^{\infty} \beta^j \mathbb{E}\left[\frac{\theta_{t+j}}{C_{t+j}^{\mathrm{B}}}\right] \right)^{-1} + \left(x_t - x_{t-1}^*\right)/l} \\
&= \frac{2w\left[K_{t-1}^*\right]}{M_t + \left(x_t - x_{t-1}^*\right)/l}.
\end{aligned}$$

Compare to the steady state price gap, one can easily prove that

$$M_t := \left(\sum_{j=1}^{\infty} \beta^j \mathbb{E}\left[\frac{\theta_{t+j}}{C_{t+j}^{\mathrm{B}}}\right] \right)^{-1} < M^*$$

from HH's budget constraints. As $x_t - x^* < 0$, the immediate price spread $\phi_t > \phi^*$.

The long-term effects follow naturally from Proposition II.5.

□

II.A.11 Proof of Corollary II.2

Proof. We start the proving by comparing with two alternative thought scenarios.

(1) The same shock hits the economy at the unhealthy steady state (K_u^*, x_u^*, ϕ_u^*) and the after-shock market expectation is to revert to the same pre-shock state.

(2) The same shock hits the economy at the healthy steady state (K_h^*, x_h^*, ϕ_h^*) and the after-shock market expectation is to revert to the same pre-shock state.

From Corollary II.1, we know IM's obligated cash outflow immediately after the shock in both scenarios are $x_u^*\phi_{u,t}$ and $x_h^*\phi_{h,t}$. In addition, $\phi_{u,t} > \phi_{h,t}$, which follows from the pricing formulas of ϕ_t and Proposition II.6.

Now consider the scenario described in Corollary II.2, that is, starting from the healthy steady state and head to the unhealthy one. Compared with Scenario 2, we can derive $\phi_t > \phi_{u,t}$. This is because the corollary scenario only differs in the pre-shock position size x_h^*. Even for the same after-shock price $\phi_{u,t}$, IM will incur larger obligated cash outflow $x_h^*\phi_{u,t} > x_u^*\phi_{u,t}$. Thus, IM's wealth loss get amplified more compared to Scenario 2 case. From the proof of Corollary II.1, K_t and x_t drops more: $K_t < K_{u,t}$, $x_t < x_{u,t}$ and $\phi_t > \phi_{u,t}$.

Similarly, compared with Scenario 1, because $\phi_t > \phi_{u,t} > \phi_{h,t}$, the obligated cash outflow at t is larger, $x_h^*\phi_t > x_h^*\phi_{u,t}$. Thus, K_t and x_t drops more: $K_t < K_{h,t}$, $x_t < x_{u,t} < x_{h,t}$.

The feature of the long-term convergence follows from Proposition II.5.

□

II.A.12 Proof of Corollary II.3

Proof. When the economy is expected to move towards an unhealthy new steady state, where the long-run position size $x_1^* < x^*$ and price spread $\phi_1^* > \phi^*$, the immediate effects follow from the analysis in the proof of Corollary II.4.

If the economy switches to a healthy new steady state, where the long-run position size $x_2^* > x^*$ and price spread $\phi_2^* < \phi^*$, the immediate reaction is that the price gap ϕ_t decreases. This can be traced back from the pricing formula with the lower ϕ_2^* as terminal point, when we assume all other variables stay the same. Thus, IM have less than expected cash outflow at t, or increase in wealth, e.g. $W_t > W^*$. Hence, this induces IM to increase capital investment $K_t > K^*$. To smooth the consumption, IM will only gradually increase the liquidity supply over time, e.g. $x_t < x_{t+1} < \ldots$. Both K_t and x_t in turn result in $\phi_t < \phi^*$.

In the long run, following from Proposition II.5 and Proposition II.7, IM will gradually reduce capital investment to K^*, increases liquidity supply to the new steady state level $x_t < x_{t+1} < \ldots < x_2^*$. In the meantime, price gaps also falls to ϕ_2^*.

□

II.A.13 Proof of Corollary II.4

Proof. From Proposition II.7, it follows that the cutoff value for IM's discount factor $\bar{\rho}$ increases after u jumps.

If $\rho > \bar{\rho}_1$, IM are very patient and their collateral constraints are slack in both pre-shock and after-shock steady states. That is, in the long run IM provide full liquidity to markets and $K^* = K_s^*$, $\phi^* = 0$ in both cases. From Proposition II.2, we know the model resembles the neoclassical growth model with frictionless markets for very patient IM. Thus, before and after the shock, IM provide full liquidity and any price gaps are fully eliminated. Therefore, $u_t = u_1$ and $\phi_t = \phi^* = 0$. Both IM's wealth and capital investment stay the same.

If $\bar{\rho} < \rho \leq \bar{\rho}_1$, IM are expected to switch from a slack state to a binding one. IM carry position size $x^* = u$ from previous period. Similar to the analysis in the Corollary II.2, the expectation of long-run positive price gap increases the immediate price gap $\phi_t > 0$. Therefore, IM suffer unexpected losses in wealth through their now positive obligated cash outflow and must reduce their capital investment $K_t < K^*$ and liquidity supply $x_t < u$. Especially when the market expectation of the new steady state is the unhealthy one, the price gap could increases so dramatically that the cash outflow $u\phi_t$ exceeds the total production income, causing IM's bankruptcy. The long-run recovery path follows directly from Proposition II.6 and Proposition II.7.

□

II.A.14 Proof of Corollary II.5

Proof. From Proposition II.2, after the shock if $\rho > \bar{\rho}_2$, IM will immediately close up all price gaps by providing full liquidity, i.e. $x_t = u_2$. However, if $\rho > \bar{\rho}$, the price spreads stay zero before and after shock, IM's wealth and thus capital investment are not affected by the shock.

If $\bar{\rho}_2 < \rho \leq \bar{\rho}$, the price spreads are positive before the shock and after the shock collapse to zero. Thus, IM's obligated cash outflows in t reduce to zero. IM gain unexpected wealth through reduced obligation. After shock, their capital investment will be the same as in the neoclassical models. With increased wealth, they will raise their capital investment immediately, i.e. $K_t > K^* > K_s^*$. From Proposition II.5 and Proposition II.7, in the long run, IM's capital will gradually falls to the new steady state level K_s^*.

□

II.A.15 Proof of Corollary II.6

Proof. The proof of short-run effects when starting from a healthy steady state $(K^*,\ x_2^*,\ \phi_2^*)$ is in the same argument as the proof of the first part of Corollary II.4.

Similar to the proof of Corollary II.5, as the new steady state price spread $\phi'^* < \phi^*$, $K^* = K'^*$ and $x'^* > x^*$, $\phi_t < \phi^*$. Thus, IM's obligated cash outflow reduces unexpectedly and this increases their wealth. As a result, $K_t > K^*$ and $x_t > x^*$. The collateral constraints keep slack until IM reach the new steady state.

The long-run effects follow from Proposition II.5 and Proposition II.7.

□

Chapter III

Risky Arbitrage and Collateral Policies

We construct a dynamic model economy in which investors from segmented markets have varying financial asset demands. Intermediaries make arbitrage profits by exploiting the price spreads across markets. Meanwhile, they are required to separately post collateral to support arbitrage trades. We show that with volatile asset demands, arbitrage becomes risky. With information friction, a looser collateral policy might render the economy more vulnerable to extremely large demand shocks, while a tighter collateral constraint helps maintain the stability at the cost of market liquidity supply.

Keywords: collateral constraints, limit of arbitrage, market liquidity, segmented markets

JEL Classification: D52 D58 G01 G12

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III.1 Introduction

This paper is a theoretical study into how financial arbitrage interacts with the aggregate economic activities through collateral constraints. For an economy in which collateral constraints are endogenously determined, we examine the limits of arbitrage and the persistence of price discrepancies between identical financial assets in segmented markets. Especially, we ask how the risk arising from volatile asset demands might affect the market liquidity, asset prices and aggregate durable goods investment. Also, we investigate how inaccurate foresight on the market demand might trigger systemic risk, in the sense that the negative impacts in the financial markets spill over to the real sector. We are centrally interested in exploring under which collateral policy the economy becomes more resilient in the event of various unanticipated demand shocks.

For this purpose, we consider a dynamic production economy in which households from two segmented markets have opposite asset demands, but can only trade with each other indirectly through competitive intermediaries. The opposite asset demands lead to price discrepancies between identical assets in two markets and create potential arbitrage opportunities for the intermediaries. While the intermediaries can profit from exploiting the price differences, they also face separate collateral constraints in each market. Such constraints arise naturally because the households cannot force intermediaries to honor the contracts unless their asset positions are secured. In our setting, intermediaries' durable goods investment plays a dual role: not only as capital for production, but also serves as collateral for trading assets. Thus, the endogenous collateral constraints limit intermediaries' asset positions as a function of their collateral holdings. In this way, intermediaries' capital investment affects and is affected by the arbitrage trades. This dynamic interaction turns out to be a powerful transmission channel by which the effects of information friction persist, amplify and spread out.

In our baseline model, we assume that all agents have perfect foresight of future asset demand shocks. With this assumption, intermediaries' arbitrage is essentially riskless: while each of the legs of an arbitrage trade is risky, the risks offset each other across markets. We derive analytically the model dynamics and the steady states of the market liquidity, asset prices as well as the aggregate capital investment. Given a unique steady state, the economy exhibits the self-recovery feature after being hit by unexpected shocks.

Later we extend our model by introducing volatile market demands. As intermediaries do not have perfect foresight of the future market demand, their arbitrage positions now become risky: the future price spread is no longer deterministic and might move against them. As intermediaries make arbitrage profits by speculating on the convergence of the future price wedge, they might incur heavy losses if the spread instead further widens up. To investigate intermediaries' trading strategy in the presence of demand risk, we first assume they have perfect information of the underlying distribution of future market demands. Compared to the riskless arbitrage case, intermediaries tend to be more aggressive in trading when the market demand and the resulting arbitrage profitability are low, and more conservative if otherwise. This is because demand risk helps enlarge the price spread and increase the arbitrage profitability for intermediaries. When the market demand is low, the endogenous collateral constraint becomes looser, allowing them to trade more. However, when the arbitrage profitability is higher, the resulting collateral constraint tightens up and becomes more inhibitive for intermediaries to expand their volumes.

To draw more welfare implications of the risky arbitrage, we also investigate the impacts of different collateral policies on intermediaries' arbitrage trading and capital investment. We define the tightness of collateral policies by the value-at-risk (VAR) that the collateral constraints impose on future loss recovery from default. We find that in general, a looser collateral policy features higher market liquidity supply and benefits household investors with more risk sharing. However, since intermediaries cannot internalize the impact of their trading volume on prices, their arbitrage profitability is relatively lower compared to that under a tighter collateral policy. Accordingly, there is also less aggregate capital investment in the economy due to the income effect. In contrast, a tighter collateral constraint induces higher trading profitability and more capital expenses at the cost of market liquidity supply.

Furthermore, we extend our analysis by introducing information friction, in the sense that agents do not have perfect information of the underlying risk and are subject to unanticipated demand shocks. Within this framework, we compare the shock responses under different collateral policies. We show that without knowing the possibility of the extreme shocks, intermediaries tend to overinvest in the arbitrage trades. This allows for more liquid financial markets and less mispricing among assets. However, it also renders the economy vulnerable to systemic risk when some low-probability, extreme shocks occur. Comparably, intermediaries under a looser collateral policy get hit more severely in the wake of a surprisingly large demand shock. This is because the suddenly widened price spread amplifies the losses of their overinvestment. As a result, the following financial distress spills over to the real sector and causes contractions. However, when the market demand is unexpectedly low, they are relatively better off. The resulting narrower price spread alleviates the negative consequence of the overinvestment. In contrast, a tighter collateral policy helps curb intermediaries' overinvestment tendency and better maintain stability in the event of unanticipated huge shocks.

The contribution of our paper is to set up a model linking risky arbitrage and durable goods investment, and provide a new prospective to study arbitrage trading strategies and collateral policies. Our paper complements a growing literature on the limits of arbitrage, and especially to its strand stressing arbitrageurs' collateral constraints. We contribute to this literature by integrating the arbitrage trading into a macroeconomic model. Our setup of arbitrage trading borrows heavily from Gromb and Vayanos (2002, 2017). There they develop a general equilibrium model in which collateral constrained arbitrageurs intermediate trade across segmented markets. Our main departure from these models is that we allow for a broader range of assets (as opposed to only the riskless asset) to serve as collateral in the financial market. This enables us to study the spillover effects between the financial and real sector. Also, rather than stressing the self-correcting feature of the riskless arbitrage as in Gromb and Vayanos (2017), our model emphasizes the risky arbitrage in the presence of volatile asset demands and information friction. We also compare the robustness of the economy under various collateral policies towards unanticipated demand shocks, as in the crisis times.

Our paper also shares similarities with many other models featuring financially constrained arbitrageurs. To name a few, Shleifer and Vishny (1997) are the first to study how trading restrictions may affect arbitrageurs' capability to correct mispricing. Due to frictions arising from the asymmetric information and moral hazard, arbitrageurs bear insolvency risk related to the margin requirement. Brunnermeier and Pedersen (2009) study the feedback loops of arbitrageurs' funding liquidity and market liquidity, and how they interact through the collateral constraints. The funding liquidity in their model captures the arbitrageurs' capability of raising debt to facilitate the arbitrage trading. Comparably, our principle difference lies in the source and objective of arbitrageurs' funding. In our model, the major funding comes from arbitrage profits rather than from direct borrowing, and it is reinvested into capital and consumptions. Hence, the funding liquidity here is reflected by the market liquidity of financial assets. In He and Krishnamurthy (2012, 2013), arbitrageurs can raise funds from other investors to invest in a risky financial security, but this external funding cannot exceed a given ratio of their own wealth. Liu and Longstaff (2004) derive the optimal arbitrage strategy of risk-averse, collateral-constrained arbitrageurs in a partial equilibrium. We differ from them by demonstrating that due to the price externality, arbitrageurs tend to undertake countercyclical trading strategies in the wake of volatile demands. Xiong (2001) and Kyle and Xiong (2001) examine the impact of arbitrage capital on asset prices by studying the wealth effects of arbitrageurs with log utility in a continuous - time model.

Our model is also related to a fast growing literature focusing on the impact of collateral constraints as financial frictions on asset pricing and aggregate economic activities. In general, there are two main approaches of modeling collateral constraints. The first approach generally assumes that there is no external enforcement to prevent potential default. Accordingly, lenders or other collateral receiving parties have to take into account of the future default possibility ex ante, based on which they impose the specific collateral requirements. In these models, the collateral constraints often limit the current borrowing or trading activities as a function of future asset prices. Examples are Kiyotaki and Moore (1997), Kübler and Schmedders (2003), Chien and Lustig (2010). The other approach assumes implicitly that there is no default in the economy. They model the collateral constraints in terms of current asset prices and assume implicitly that agents would never walk away from their liabilities. In these models, agents do not have to calculate or estimate future asset prices. This allows for more flexibility in modeling shocks and uncertainty. For instance, Mendoza (2010), Bianchi (2011) and Schmitt-Grohé and Uribe (2016) model the endogenous borrowing constraints with the current relative prices. Our model belongs to the first approach and assumes limited liability in case of default.

III.2 Baseline Model

Figure III.1 shows the basic structure of our model. We consider a discrete time, infinite horizon economy with two segmented markets, market A and B. There are two types of competitive agents: household investors within each single market and specialized intermediaries. Each constitute a continuum of unit measure.

Within each segmented market, there exists an identical, long-lived risky financial asset, which pays out dividends in every period and is each in zero net supply. We identify them in different markets as asset A and B. Households in market A (B) can only trade asset A (B). As will become clear later, households' demands for the risky asset is opposite in market A and B. Intermediaries can exploit arbitrage profits by speculating on the price spreads between segmented markets.

III.2.1 Households

We assume households in each market are competitive, live infinitely and share the same preference. Each household receives natural endowments in every period, which follow an exogenous process of the form

$$e_t^i = h_t + u_{t-1}^i \theta_t, \quad i \in \{\mathrm{A}, \mathrm{B}\},$$

where e_t^i is the endowment of one household in period t and market i, h_t is a constant and $u_{t-1}^i \theta_t$ is the endowment shock. In particular, $\{\theta_t\}_{t=1}^{\infty}$ is a sequence of independent identically distributed random variables, each of which follows a symmetric distribution around zero on a bounded support $\mathcal{S} = \left[-\bar{\theta}, \bar{\theta}\right]$, where $\bar{\theta} > 0$. We refer to θ_t as the shock unit. u_{t-1}^i is the random shock intensity and is independent of $\{\theta_t\}$. It is always revealed one period ahead, so households know their hedging demand in advance.

We assume that the shock intensities in the two markets are identical in magnitude but opposite in their signs:

$$u_t^{\mathrm{A}} = -u_t^{\mathrm{B}} > 0.$$

This setup provides a simple way to motivate the real-world price spreads between similar assets from the diverse demands in segmented markets. Without further intermediation, the volatilities of the consumption paths in the two markets are perfectly negatively correlated. Thus, households from the two markets have opposite hedging demands.

Households' expected utility is given by

$$\mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s \ln \left(c_{t+s}^i \right) \right], \quad i \in \{\mathrm{A}, \mathrm{B}\},$$

where c_{t+s}^{i} is households' $t+s$ consumption in market i, $i \in \{\mathrm{A}, \mathrm{B}\}$, and $0 < \beta < 1$. To ensure non-negative consumptions for households, we set $h_t u_{t-1} \bar{\theta} > 0$ for all t.

III.2.2 Financial Assets

Within each market, there is an identical financial asset. It is long-lived, in zero-net supply and pays out a dividend equal to θ_t in t. As the dividend mirrors the shock unit perfectly, it serves as an ideal hedging instrument for households.

Since the hedging demands in the two markets are opposite, the asset price, i.e. P_t^i, $i \in \{\mathrm{A}, \mathrm{B}\}$, would differ if there were no intermediation. As households in market A always encounter a positive amount of shock units, i.e., $u_t = u_t^{\mathrm{A}} > 0$, they are inclined to sell the assets to neutralize their endowment shocks. To the extent that market A has negative asset demands while market B has the opposite, prices in market A tend to be lower than in market B. We define the price difference between the two markets as

$$\phi_t := P_t^{\mathrm{B}} - P_t^{\mathrm{A}}.$$

III.2.3 Intermediaries

Outside the two markets, there is a continuum of measure one of competitive, homogeneous and infinitely lived intermediaries. Unlike household investors, they can trade financial assets simultaneously in both markets. The price spreads create potential arbitrage opportunities for them. They can earn profits by entering long positions in the low-price market and taking short ones in the high-price region. By doing so, they also provide market liquidity to households in both markets. To simplify analysis, we assume further that intermediaries will incur inhibitive cost if they do not take net zero positions. Consequently, their positions in the two markets must be identical in magnitude yet opposite in the signs. Denote x_t as their positions in market A

$$x_t := x_t^{\mathrm{A}} = -x_t^{\mathrm{B}}.$$

As in Gromb and Vayanos (2002, 2017), we use x_t as a measure of market liquidity.

In addition to intermediating asset demands between markets, intermediaries also play an important role in aggregate production by investing capital as entrepreneurs. For simplicity, we assume they have the unique capability to convert the perishable consumption goods one-to-one into physical capital and vice versa. Accordingly, they organize the production sector in the economy and receive income from production in period $t+1$ by investing K_t units of capital at t:

$$y_{t+1} = Z_{t+1}K_t + (1-\delta)K_t, \tag{III.1}$$

where Z_{t+1} is the productivity factor and $\delta \in [0,1]$ is the depreciation rate. $Z_{t+1}K_t$ is the production output and $(1-\delta)K_t$ is the residual capital after depreciation.

Their expected utility is given by

$$\mathbb{E}_t \left[\sum_{s=0}^{\infty} \rho^s \ln \left(c_{t+s}^{\mathrm{IM}} \right) \right],$$

where c_{t+s}^{IM} is IM's $t+s$ consumption and $0 < \rho < 1$.

The assumptions regarding intermediaries aim to capture the sophisticated investors in reality, such as hedge funds and investment banks with wide investment spectrum. Those investors are less constrained by regulatory restrictions or informational asymmetry that lead to potential market segmentation. For instance, some international hedge funds diversify their investment across sectors ranging from high-tech start-ups to stock markets across borders.

III.2.4 Collateral Constraint

The financial friction in this model comes from the intermediaries' collateral constraints when they engage in arbitrage trade. The constraints arise from the long-lived feature of the financial assets. In contrast to the one-period contracts, in which asset prices collapse to zero in the next period, the positions of long-lived securities remain alive and carry value in all future periods. The sequential trading of these assets thus obliges intermediaries to repay the liability from their previous positions before taking new ones in both markets. As will become clear, such liquidation of previous arbitrage positions usually involves some due payments to households.

In order to ensure that intermediaries honor their contracts later, households require them to deposit collateral up to the amount that they would not have an incentive to walk away in the next period. Intermediaries are requested to do so market by market and cannot use positions in one market to collateralize trades in another, due to the segmentation. That is, they have to separately post collateral to cover the maximal

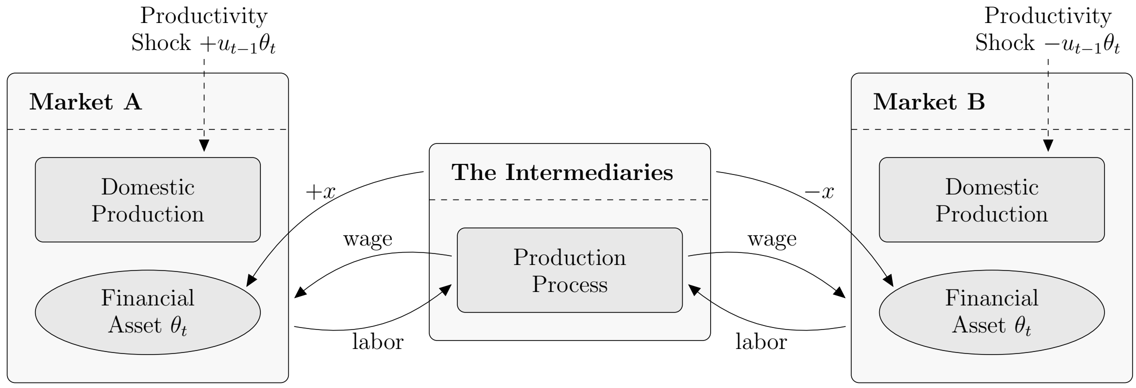

Figure III.1: The structure of the economic system.

potential future loss in each market. To be consistent with the real-world liquidation as well as the literature of limited liability, we further assume that in case of default households can only grab intermediaries' depreciated capital, without being able to confiscate their production output.

Denote x_t^i as intermediaries' position in asset i and K_t as her total physical investment. The total value of the intermediaries' margin account in both market in period $t+1$ has to satisfy

$$\sum_{i \in \{\mathrm{A,B}\}} \left(x_t^i P_{t+1}^i + \min_{\theta_{t+1}} \left\{ x_t^i \theta_{t+1} \right\} \right) + (1-\delta)K_t \geq 0. \tag{III.2}$$

With this, we can imply that the collateral constraints become more restrictive when the dividend become more volatile, i.e., $\bar{\theta}$ is higher. This is because intermediaries have to put more collateral per unit of position to insure households. Also, if the price gap at $t+1$ is larger, more capital is required, as this is positively related to the future liability arising from current positions. This particular form differs with the one in Zhang (2017). Here, intermediaries have to collateralize for their maximal liability separately in each market, while they only have to prepare to cover their actual liability in Zhang (2017). Thus, the collateral constraints in the present model is strictly tighter.

In this way, the collateral constraint limits the intermediaries' positions and their ability to exploit arbitrage opportunities as a function of their capital investment. As a result, the durable goods investment derives value not only from production, but also from its role serving as collateral in the financial markets.

III.2.5 Optimization Problems

The intermediaries' optimization problem is to maximize their utility by choosing the optimal level of consumption c_t^{IM}, physical investment K_t and positions x_t^i in both financial markets:

$$\max_{c_s^{\mathrm{IM}}, x_s^i, K_s} \mathbb{E}_t \left[\sum_{s=t+1}^{\infty} \rho^{s-t} \ln \left(c_s^{\mathrm{IM}} \right) \right], \quad i \in \{\mathrm{A}, \mathrm{B}\}.$$

subject to the budget constraint

$$\begin{aligned} c_t^{\mathrm{IM}} &= \underbrace{\sum_{i \in \{\mathrm{A},\mathrm{B}\}} x_{t-1}^i P_t^i}_{\substack{\text{value of previous} \\ \text{period's positions}}} - \underbrace{\sum_{i \in \{\mathrm{A},\mathrm{B}\}} x_t^i P_t^i}_{\substack{\text{immediate proceeds} \\ \text{from taking} \\ \text{new positions}}} + \underbrace{Z_t K_{t-1} + (1-\delta) K_{t-1} - K_t}_{\text{production income}} \\ &= (x_t - x_{t-1}) \phi_t + Z_t K_{t-1} + (1-\delta) K_{t-1} - K_t, \end{aligned}$$

and the following collateral constraint

$$\sum_{i \in \{\mathrm{A},\mathrm{B}\}} \left(x_t^i P_{t+1}^i + \min_{\theta_{t+1}} \left\{ x_t^i \theta_{t+1} \right\} \right) + (1-\delta) K_t \geq 0.$$

Or put simply,

$$(1-\delta) K_t \geq x_t \left(\phi_{t+1} + 2\bar{\theta} \right). \tag{III.3}$$

The corresponding transversality condition is

$$\lim_{t \to \infty} \rho^{t+1} \mathbb{E} \left[\frac{(Z_t + 1 - \delta) K_t}{c_{t+1}^{\mathrm{IM}}} \right] = 0.$$
$$\lim_{t \to \infty} \rho^{t+1} \mathbb{E} \left[\frac{\phi_{t+1} x_t}{c_{t+1}^{\mathrm{IM}}} \right] = 0.$$

Compared to the intermediaries, households are not restricted by the collateral constraint. Their optimization problem is given by

$$\max_{c_s^i, y_s^i} \mathbb{E} \left[\sum_{s=t}^{\infty} \beta^{s-t} \ln \left(c_s^i \right) \right]$$

subject to the dynamic budget constraint

$$c_t^i = h + \left(y_{t-1}^i - y_t^i\right) P_t^i + \left(y_{t-1}^i + u_{t-1}^i\right) \theta_t, \quad \text{for } i \in \{\mathrm{A}, \mathrm{B}\}.$$

Ideally, given enough liquidity supply, the households would like to take a position size of $y_t^i = -u_t^i$ to eliminate all the consumption risk arising from θ_t.

Likewise, the corresponding transversality condition is

$$\lim_{t \to \infty} \beta^t \mathbb{E}\left[\frac{P_t^i y_t^i}{c_t^i}\right] = 0.$$

III.2.6 Sequential Equilibrium

Given the initial capital investment K_0 and agents' asset positions x_0 and y_0^i, $i \in \{\mathrm{A}, \mathrm{B}\}$, an equilibrium is described by the price process P_t^i, capital investment K_t, asset holdings y_t^i and x_t^i, and consumption choices C_t^{IM} and C_t^i such that

- all agents solve their optimization problems given prices;
- markets clear for financial assets, that is $y_t^i + x_t^i = 0$.

III.3 Riskless Arbitrage

To illustrate how the model works, we first characterize the equilibrium under a deterministic shock intensity $\{u_t^i\}$. Within this setting, the only uncertainty thus arises from the shock unit θ_t. Since intermediaries hold zero net positions of financial assets, their capital investment and arbitrage trading decisions are unaffected by the realization of θ_t. As will become clear later, under this condition, intermediaries' arbitrage trades are riskless and their optimization problems reduce to be deterministic.

As the technology is a linear function of the capital input, we make use of this simple form and construct the following conjecture of intermediaries' value function:

$$V_t^{\mathrm{IM}}\left(K_{t-1}, x_{t-1}\right) = C \ln\left(\left(Z_t + 1 - \delta\right) K_{t-1} - x_{t-1}\phi_t\right) + D_t, \tag{III.4}$$

where C is a constant and D_t is a deterministic function of t.

Proposition III.1. *The value function of an intermediary in period t is given by (III.4), where $C = \rho/(1-\rho)$. The intermediary's optimal consumption is*

$$c_t^{IM} = (1-\rho)\left[(Z_t + 1 - \delta) K_{t-1} - x_{t-1}\phi_t\right] = (1-\rho)W_t, \tag{III.5}$$

where

$$W_t := (Z_t + 1 - \delta) K_{t-1} - x_{t-1}\phi_t$$

is the intermediary's wealth at the beginning of t.

This proposition shows that intermediaries are myopic as they will always consume a fixed portion of their total wealth, independent of the wealth level and investment returns.

Unlike the intermediaries, the households' still face uncertainty from the shock unit θ_t, even when $\{u_t\}$ is deterministic. From their budget constraint, we can see that the households' exposure to θ_{t+1} is the sum of their asset position y_t^i and the shock intensity u_t. In particular, we conjecture that under the certainty case, the households' value function in period t is

$$V_{i,t}\left(y_t^i\right) = \mathbb{E}_t\left[O \ln\left(y_t^i + Q_t^i\right) + N_t^i\right] \tag{III.6}$$

where O is a constant, Q_t^i and N_t^i are deterministic functions of the next period's shock θ_{t+1} and price level P_{t+1}^i.

Proposition III.2. *The value function of an household i, $i \in \{A, B\}$, in period t is given by (III.6), where $O = \beta/(1-\beta)$.*

First, we look at the equilibrium price difference.

Lemma III.1. *Define $P_t^A(\theta_t)$ and $P_t^B(\theta_t)$ to be the time t equilibrium prices in market A and B as functions of θ_t. It follows that*

$$P_t^A(\varepsilon) = -P_t^B(-\varepsilon),$$

where $\varepsilon \in \left[-\bar{\theta}, \bar{\theta}\right]$.

Proposition III.3. *In equilibrium, the asset prices are given by*

$$P_t^A = -\frac{c_t^A}{c_t^A + c_t^B}\phi_t = -\left(\frac{1}{2} + \frac{(u - x_{t-1})\,\theta_t}{2h}\right)\phi_t, \tag{III.7}$$

$$P_t^B = \frac{c_t^B}{c_t^A + c_t^B}\phi_t = \left(\frac{1}{2} - \frac{(u - x_{t-1})\,\theta_t}{2h}\right)\phi_t, \tag{III.8}$$

and the price difference

$$\phi_t = \frac{2h}{M_t + (x_t - x_{t-1})}, \tag{III.9}$$

where

$$M_t := \left(\mathbb{E}_t\left[\sum_{j=1}^{\infty}\beta^j\frac{\theta_{t+j}}{c_{t+j}^B}\right]\right)^{-1}.$$

Moreover, ϕ_t is independent of the realization of shock units θ_t, $\forall t$.

Given the price difference ϕ_t, the price in i is proportional to market i's consumption share relative to households' total endowment in both markets. Specifically, the proportion depends on the realization of θ_t. The prices in both market decrease with θ_t. However, the price difference ϕ_t, as a whole, is independent of the specific realization of θ_t.

Intuitively, since intermediaries take a zero net position in the financial assets, their consumptions are independent of the realization of θ_t. Technically, from households' first order conditions, we make use of the logarithmic utility and the symmetric distribution of θ_t. We further find that the independence of ϕ_t on θ_t also holds for the cases where households have more general CRRA utility and θ_t follows a two-point distribution.

III.3.1 Dynamics of Intermediaries' Capital

With deterministic shock intensity and the myopic feature of intermediaries, we can derive the dynamics of their capital growth.

Proposition III.4. *Intermediaries' capital grows according to*

$$K_t = \rho G_t K_{t-1}, \tag{III.10}$$

where G_t is the growth rate.

- *If*

$$K_{t-1} \geq \frac{2u\bar{\theta}}{\rho(1-\delta)\left(Z_t + 1 - \delta\right)},$$

then the collateral constraints at t are slack, with $G_t = Z_t + 1 - \delta$. Intermediaries only receive income from production.

- *Else if*

$$K_{t-1} < \frac{2u\bar{\theta}}{\rho(1-\delta)\left(Z_t + 1 - \delta\right)},$$

then intermediaries' collateral constraints are binding, with

$$G_t = \frac{Z_t + 1 - \delta - \frac{(1-\delta)\phi_t}{\phi_t + 2\bar{\theta}}}{1 - \frac{(1-\delta)\phi_t}{\phi_{t+1} + 2\bar{\theta}}} > Z_t + 1 - \delta.$$

Intermediaries capital growth rate is accelerated by their arbitrage activities.

When intermediaries' initial capital exceeds the threshold value defined in Proposition III.4, then they have enough collateral to eliminate any price discrepancies and thus can provide full liquidity to households in both sides. However, this also means the extinction of arbitrage opportunities for intermediaries. Consequently, intermediaries only receive income from the production sector in this case.

On the other hand, when intermediaries' financial constraints are binding, they cannot absorb all the price spreads and only provide partial liquidity to both markets. In this case, they are able to earn positive arbitrage income through exploiting the current price spreads. They only have to repay a lower amount of liability in the next period, as the price gap dwindles. By reinvesting the proceeds to the durable goods, they are able to expand the production. In this sense, arbitrage opportunities offer intermediaries a channel to obtain external funding with a negative interest rate to leverage up their original savings. The higher the current price spread ϕ_t is, the more leverage intermediaries can attain from arbitrage. As intermediaries' capital investment increases, so does their capability to provide more liquidity. In turn, the price gaps diminish gradually and the capital growth also slows down accordingly.

Proposition III.5. *When the shock intensity u_t is deterministic, a competitive equilibrium exists in which the price spread ϕ_t, intermediaries' capital investment K_t and the positions of the financial assets x_t^i are deterministic.*

III.3.2 Steady State

We now consider a stationary version of the model, with $Z_t = Z$, $h_t = h$ and $u_t = u$.

Proposition III.6. *Given the shock intensity u_t and productivity factor Z_t are constant, i.e., $u_t = u$ and $Z_t = Z$ for all $t = 0, 1, 2, \ldots$,*

- *if $\rho(Z + 1 - \delta) > 1$, K_t increases towards $K = \infty$, x_t converges to $x^* = u$, and ϕ_t decreases to $\phi^* = 0$.*
- *If $\rho(Z + 1 - \delta) = 1$,*
  - *if the initial capital is $K_0 \geq 2u\bar{\theta}/(1-\delta)$, then K_t stays at K_0, $x_t = u$, and $\phi_t = 0$;*
  - *else if $K_0 < 2u\bar{\theta}/(1-\delta)$, then K_t increases towards $K^* = 2u\bar{\theta}/(1-\delta)$, x_t converges to $x^* = u$ and ϕ_t decreases to $\phi^* = 0$.*
- *Else if $\rho\bar{G} < 1$, where*

$$\bar{G} := \max_s \left\{ \frac{Z + 1 - \delta - \frac{(1-\delta)\phi_s}{\phi_s + 2\bar{\theta}}}{1 - \frac{(1-\delta)\phi_s}{\phi_{s+1} + 2\bar{\theta}}} \right\},$$

  then both K_t and x_t decrease to zero, and ϕ_t increases towards

$$\phi^* = 2h \frac{\beta}{1-\beta} \mathbb{E}_t \left[\frac{\theta_{t+1}}{h - u\theta_{t+1}} \right]$$

- *Otherwise, ϕ_t converges to*

$$\phi^* = 2 \frac{1 - \rho(Z + 1 - \delta)}{\rho Z - \delta} \bar{\theta}, \tag{III.11}$$

x_t and K_t converge to values x^* and K^*, with $x^* < u$ being the solution to

$$\phi^* = 2h\frac{\beta}{1-\beta}\mathbb{E}_t\left[\frac{\theta_{t+1}}{h+(x^*-u)\,\theta_{t+1}}\right],$$

and $K^* = x^*\left(\phi^* + 2\bar{\theta}\right)/(1-\delta)$.

If the intermediaries are extremely patient, their capital and wealth will explode over time. During this process, they provide full liquidity to both financial markets and close up any price discrepancies.

In the less patient case, i.e., $\rho(Z+1-\delta) = 1$, the steady-state level of capital is also sufficient to support a slack collateral constraint, i.e., $K^* \geq 2u\bar{\theta}/(1-\delta)$. Also, the price discrepancy eventually diminishes to zero and intermediaries provide full liquidity to both markets, i.e., $x_t = u$. This is because any non-zero price gap would induce unconstrained intermediaries to increase their position x^* in the financial assets for more arbitrage profit. As a result, similar to the previous scenario, intermediaries receive income solely from production.

However, when intermediaries are not patient enough, they do not have the tendency to invest much capital over time to maintain slack collateral constraints. For extremely impatient intermediaries, i.e., $\rho\bar{G} < 1$, even with the relatively high arbitrage proceeds compared to their own savings, they still cannot maintain the level of their previous capital investment. Hence, their capital stocks keep diminishing until zero. The same happens for their position sizes in both markets. Eventually, the price disparage would converge to the autarky case.

For the not so extreme type, intermediaries are able to reach a steady level of capital investment, consumption and partial liquidity supply, i.e., $x^* < u$. The collateral constraints are binding in the steady state. In this scenario, intermediaries' arbitrage income $x_t\phi_t$ in each period exactly offsets with their liability $x_{t-1}\phi_t$. One can regard the arbitrage income as a nominal zero-interest loan borrowed from households by intermediaries to leverage up their capital investment. As intermediaries can always roll over the old contracts with the same amount of new loans, they would never have to repay their initial loans. Thus, intermediaries' arbitrage trading is indeed riskless.

The pricing expression, i.e., Equation (III.11), shows that the steady state price gap decreases with intermediaries' time preference ρ^* and productivity Z, while it increases with the depreciation rate δ and the volatility parameter of the endowment shocks, i.e., $\bar{\theta}$.

III.4 Risky Arbitrage

Having explored the riskless arbitrage in which $u_t = u$, we now turn our attention to the more general cases with random shock intensity series $\{u_t\}$. One of the key differences is that intermediaries' optimization problem is no longer a deterministic one. Rather, intermediaries now face uncertain returns of their arbitrage trading, as the resulting liability arising from settlement depends on the realization of the shock intensity in the next period. In this sense, the previous described arbitrage trading now becomes risky.

In this context, we aim to study intermediaries' trading strategy in the wake of demand risks. Also, we are interested in examining how different policies on collateral constraints may affect the overall welfare, aggregate output, as well as intermediaries' capability to eliminate mispricing and provide liquidity. In particular, different policies on collateral constraints in our model impose different value-at-risk (VAR) control on intermediaries' trading positions. We further investigate how they might affect the overall market stability and real activities in the presence of unanticipated demand shocks.

For this purpose, we conduct two numerical exercises. In the first experiment, we assume that all agents have perfect information of the underlying distribution of the shock intensity $\{u_t\}$ and that there are no unanticipated shocks. We compare the welfare implications, capital accumulation, market liquidity as well as price spreads under collateral constraints of varying degree of tightness, in the equilibrium paths with or without low-probability realization of u_t. In the second exercise, we include the possibility that the agents might experience some unanticipated shocks in $\{u_t\}$, and assume that they cannot quickly update their belief over the distribution of $\{u_t\}$ in a short time. This allows us to investigate whether the tighter or looser collateral constraints help stabilize the overall economy in crisis times. For example, we compare intermediaries' financial robustness towards extreme unanticipated shocks under different collateral policies.

In particular, we conduct the two thought experiments through the numerical computation of a concise four-period model. We solve the model and simulate with all possible paths of $\{u_t\}$. We then compare the dynamics of capital accumulation, market liquidity and asset prices in equilibrium. In the following, before we present our numerical examples, we will first introduce the specifics of the four-period model and define the tightness of collateral constraints in our exercise.[1]

[1] We also attach the computation of an infinite-horizon model, including the detailed recursive formation and computation algorithm, in the appendix.

III.4.1 The Four-Period Model

The four-period model inherits the same setup from the previous general model, except both the assets' and the agents' lifespan, as the name suggests, extend only four time intervals i.e., $t \in \{1,2,3,4\}$. In particular, at $t=4$ there will be no trading of the financial assets in neither market, since after this period all agents cease to exist and no households demand the assets to hedge their risk anymore. As a result, the prices converge to zero in both markets.

Thus, the intermediaries' optimization problem in the four-period model with initial physical capital K_0 and asset position x_0 can be represented as:

$$\max \left\{ \ln C_1^{\mathrm{IM}} + \rho \mathbb{E}\left[\ln C_2^{\mathrm{IM}}\right] + \rho^2 \mathbb{E}\left[\ln C_3^{\mathrm{IM}}\right] + \rho^3 \mathbb{E}\left[\ln C_4^{\mathrm{IM}}\right] \right\},$$

where the maximum is taken over $\left(C_1^{\mathrm{IM}}, C_2^{\mathrm{IM}}, C_3^{\mathrm{IM}}, C_4^{\mathrm{IM}}, K_1, K_2, K_3, x_1, x_2, x_3\right)$ subject to the budget constraints at $t \in \{1,2,3,4\}$:

$$\begin{aligned}
C_1^{\mathrm{IM}} &= \left(P_1^{\mathrm{A}} - P_1^{\mathrm{B}}\right)(x_0 - x_1) + (Z_1 + 1 - \delta) K_0 - K_1, \\
C_2^{\mathrm{IM}} &= \left(P_2^{\mathrm{A}} - P_2^{\mathrm{B}}\right)(x_1 - x_2) + (Z_2 + 1 - \delta) K_1 - K_2, \\
C_3^{\mathrm{IM}} &= \left(P_3^{\mathrm{A}} - P_3^{\mathrm{B}}\right)(x_2 - x_3) + (Z_3 + 1 - \delta) K_2 - K_3, \\
C_4^{\mathrm{IM}} &= (1 - \delta + Z_4) K_3.
\end{aligned}$$

And the financial constraints at $t \in \{1,2,3\}$:

$$x_1 \left[P_2^{\mathrm{B}}\left(u \,|\mathbb{P}\{u_2 \le u\} = \alpha\right) - P_2^{\mathrm{A}}\left(u \,|\mathbb{P}\{u_2 \le u\} = \alpha\right) + 2\bar{\theta}\right] \le (1-\delta)K_1, \tag{III.12}$$

$$x_2 \left[P_3^{\mathrm{B}}\left(u \,|\mathbb{P}\{u_3 \le u\} = \alpha\right) - P_3^{\mathrm{A}}\left(u \,|\mathbb{P}\{u_3 \le u\} = \alpha\right) + 2\bar{\theta}\right] \le (1-\delta)K_2, \tag{III.13}$$

$$2x_3\bar{\theta} \le (1-\delta)K_3. \tag{III.14}$$

Here, we assume $u_t = u_t^{\mathrm{A}} = -u_t^{\mathrm{B}} > 0$ and therefore in equilibrium $x_t = x_t^{\mathrm{A}} = -x_t^{\mathrm{B}} > 0$ for $t \in \{1,2,3,4\}$. Thus, households' optimization problem reduces to the following form.

- For Household A

$$\max \left\{ \ln C_1^{\mathrm{A}} + \beta \mathbb{E}\left[\ln C_2^{\mathrm{A}}\right] + \beta^2 \mathbb{E}\left[\ln C_3^{\mathrm{A}}\right] + \beta^3 \mathbb{E}\left[\ln C_4^{\mathrm{A}}\right] \right\}$$

where we maximize over $\left(C_1^{\mathrm{A}}, C_2^{\mathrm{A}}, C_3^{\mathrm{A}}, C_4^{\mathrm{A}}, y_1^{\mathrm{A}}, y_2^{\mathrm{A}}, y_3^{\mathrm{A}}\right)$ subject to

$$\begin{aligned}
C_1^{\mathrm{A}} &= P_1^{\mathrm{A}}(x_1 - x_0) + h + (u_0 - x_0)\,\theta_1, \\
C_2^{\mathrm{A}} &= P_2^{\mathrm{A}}(x_2 - x_1) + h + (u_1 - x_1)\,\theta_2, \\
C_3^{\mathrm{A}} &= P_3^{\mathrm{A}}(x_3 - x_2) + h + (u_2 - x_2)\,\theta_3, \\
C_4^{\mathrm{A}} &= h + (u_3 - x_3)\,\theta_4.
\end{aligned}$$

- For Household B,

$$\max \left\{ \ln C_1^{\mathrm{B}} + \beta \mathbb{E}\left[\ln C_2^{\mathrm{B}}\right] + \beta^2 \mathbb{E}\left[\ln C_3^{\mathrm{B}}\right] + \beta^3 \mathbb{E}\left[\ln C_4^{\mathrm{B}}\right] \right\},$$

where we maximize over $\left(C_1^{\mathrm{B}}, C_2^{\mathrm{B}}, C_3^{\mathrm{B}}, C_4^{\mathrm{B}}, y_1^{\mathrm{B}}, y_2^{\mathrm{B}}, y_3^{\mathrm{B}}\right)$ subject to

$$\begin{aligned}
C_1^{\mathrm{B}} &= P_1^{\mathrm{B}}(x_0 - x_1) + h - (u_0 - x_0)\,\theta_1, \\
C_2^{\mathrm{B}} &= P_2^{\mathrm{B}}(x_1 - x_2) + h - (u_1 - x_1)\,\theta_2, \\
C_3^{\mathrm{B}} &= P_3^{\mathrm{B}}(x_2 - x_3) + h - (u_2 - x_2)\,\theta_3, \\
C_4^{\mathrm{B}} &= h - (u_3 - x_3)\,\theta_4.
\end{aligned}$$

In the four-period model setup, a competitive equilibrium consists of price P_t^{A}, P_t^{B}, intermediaries' capital investment K_t and positions in the financial assets x_t for the intermediaries and y_t^{A}, y_t^{B} for the households in market A and B, for $t \in \{1, 2, 3, 4\}$, such that both intermediaries and households solve their optimization problems given prices and the markets for financial assets clear:

$$x_t + y_t^{\mathrm{A}} = 0 \quad \text{and} \quad -x_t + y_t^{\mathrm{B}} = 0, \quad \text{for } t \in \{1, 2, 3\}.$$

Tightness of Collateral Constraints

Note that the collateral constraints, i.e. the inequality condition (III.12) and (III.13), are different from those in the deterministic case. This is because unlike the latter, in the setting with random shock intensity process $\{u_t\}$, one cannot precisely determine the future price spread, and hence the amount of intermediaries' liability arising from current arbitrage positions. These can only be revealed in the next period. As a result, the ex ante collateral requirements can only specify a probability range of the next period's shock intensity, within which intermediaries must cover the maximum possible liability that their current trading positions would incur.

Put differently, the collateral requirement imposes a specific VAR control on intermediaries' positions in advance. For example, the probability range of the future shock intensity specified in the inequality condition (III.12) and (III.13) are $u_s \in [0, \alpha]$, where $0 \leq \alpha \leq 100\%$ and $s \in \{2, 3\}$. This indicates that intermediaries should post enough collateral in advance so that they can account for the due liability in case the next period's shock intensity hits the α percentile of the distribution. In this sense, the value α represents the tightness of the collateral requirement. When $\alpha = 100\%$, it imposes the tightest possible collateral constraint on intermediaries. That is, they have to cover for households' maximal potential losses for all possible realization of future shock intensity. In contrast, $\alpha = 0$ completely disregards the risk of intermediaries walking away from their previous positions and thus corresponds to a frictionless market without any collateral constraints. In the following sections, we will conduct experiments and analyze model dynamics with varying values of α to shed light on the policy implications on the collateral requirement in different settings.

In most of our following experiments, we assume collateral constraints featuring $\alpha < 100\%$. This assumption is grounded in real world practice, as most of the margin requirements or collateral constraints in reality are not designed to cover for the absolute maximum potential risk or to take into account of extreme low probability events. On one hand, both parties of the contracts in practice might not have perfect knowledge of the underlying risk or perfect foresight of all future possible events, thus they can only form contracts to cover for certain range of uncertainties based on their forecasts. On the other hand, it might be too restrictive to make any transaction possible, if one party insists on the other posting collateral against all extreme or low-probability scenarios.

Variable	u_t				θ_t	
Value	0	8	12	20	-1	1
Probability	0.01	0.49	0.49	0.01	0.5	0.5

Table III.1: Distribution of u_t and θ_t.

III.4.2 Risky Arbitrage without Unanticipated Shocks

In this section, we focus on the settings in which all agents have perfect information of the distribution of the shock intensity $\{u_t\}$ as well as the shock units $\{\theta_t\}$. Conditional on this, we aim to explore how varying degrees of tightness in collateral requirement affect the equilibrium dynamics in both "normal" periods and "abnormal" times with occurrence of low-probability events. In particular, we would like to ask which collateral requirement policy has more constructive effects on the aggregate production, market liquidity, pricing efficiency as well as overall welfare.

To achieve this, we implement simulations of two otherwise identical four-period models which only differ in the tightness of collateral constraints. We modulate the tightness of collateral constraints in the two parallel economies by assigning different values for the parameter α in the requirements (III.12) and (III.13). We then simulate the model dynamics with the same exogenous paths of the shock intensity $\{u_t\}$ and the endowment shock unit $\{\theta_t\}$.

In order to model extreme, low-probability shocks, we assume that the shock intensity $\{u_t\}$ has an i.i.d. distribution that follows a symmetric four-point distribution. For example, we model the "normal" events when u_t takes a value in 8 or 12, each having a probability of 49%. However, we also model the low-probability events when u_t takes a value in either 0 or 20, each with a probability of 1%. Also, for simplicity, we assume that θ_t for $t \in \{2, 3, 4\}$ follow a two-point distribution. In particular, θ_t take values of either $-\bar{\theta}$ or $\bar{\theta}$ with probability equal to p and $1 - p$ each. Table III.1 lists the parameter values in the distribution.

Given the above distribution, we model the first economy with a relatively loose collateral requirement by setting $\alpha = 50\%$. That is, intermediaries at t only have to post collateral to cover for the potential liability in case $u_{t+1} = 8$ happens. Similarly, we set the second economy with $\alpha = 99\%$ to illustrate a policy with tighter constraints. Table III.2 lists the common parameter values of the two economies in the experiment.

Parameter Values			
α	0.5	p	0.5
δ	0.5	x_0	0
ρ	0.9	K_0	20
β	0.9	u_0	8
$\bar{\theta}$	1	θ_0	1
θ_1	-1	u_1	8
h_t	21	Z_t	1.0

Table III.2: The set of parameter values.

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread Loose	Binding Collateral
1	8	25.706	4.8825	8.7235	0.9073	TRUE
2	12	27.095	5.9725	12.5070	0.9558	TRUE
3	8	19.822	4.9555	20.5470	0.2683	TRUE
4	0	0	0	29.7330	0	FALSE

Period	u_t	K Tight	X Tight	C^{IM} Tight	Spread Tight	Binding Collateral
1	8	25.641	4.1848	8.7235	1.0430	TRUE
2	12	26.858	5.0745	12.5500	1.0636	TRUE
3	8	19.764	4.9409	20.4880	0.2681	TRUE
4	0	0	0	29.6460	0	FALSE

Period	u_t	K Det.	X Det.	C^{IM} Det.	Spread Det.	Binding Collateral
1	8	25.5765	4.5647	8.7235	0.9420	TRUE
2	12	26.5786	5.8377	12.8066	0.8016	TRUE
3	8	19.4657	4.8664	20.1337	0.2765	TRUE
4	0	0	0	29.1986	0	FALSE

Table III.3: Model dynamics under a "normal" path of $\{u_t\}$ with random and deterministic demands.

We show the full set of model dynamic solutions with all possible paths of $\{u_t\}$ and $\{\theta_t\}$ shocks in Table III.11 in the appendix. For expositional convenience, here we only select some representatives sample paths. For example, Table III.3 shows the comparison of the key variables in equilibrium to approximate the normal time dynamics. That is, there are no low-probability events in the paths of $\{u_t\}$. Also, Table III.4 and Table III.5 illustrate the paths involving rare events, i.e., $u_t = 0$ or $u_t = 20$. As a reference, we also report the equilibrium dynamics of the deterministic case under the same sample paths of $\{u_t\}$.

From the paths of model dynamics involving rare, extreme realized values of u_t, we find that intermediaries with demand risks tend to trade relatively more when u_t is low

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread Loose	Binding Collateral
1	8	25.706	4.8825	8.7235	0.9073	TRUE
2	0	23.844	1.5715	13.648	0.3224	FALSE
3	8	18.076	4.5189	18.576	0.3003	TRUE
4	0	0	0	27.113	0	FALSE

Period	u_t	K Tight	X Tight	C^{IM} Tight	Spread Tight	Binding Collateral
1	8	25.641	4.1848	8.7235	1.0430	TRUE
2	0	23.919	1.5498	13.697	0.3213	FALSE
3	8	18.131	4.5328	18.6400	0.2990	TRUE
4	0	0	0	27.1970	0	FALSE

Period	u_t	K Det.	X Det.	C^{IM} Det.	Spread Det.	Binding Collateral
1	8	23.4178	5.2697	8.7235	0.4063	TRUE
2	0	21.6540	1.0233	12.5304	0.2219	FALSE
3	8	16.6063	4.1516	16.9160	0.3329	TRUE
4	0	0	0	24.9095	0	FALSE

Table III.4: Model dynamics of a path involving an extreme low realization of u_t with random and deterministic demands.

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread Loose	Binding Collateral
1	8	25.706	4.8825	8.7235	0.90727	TRUE
2	20	32.039	7.3092	10.944	1.8229	TRUE
3	8	23.213	5.8033	24.5560	0.19168	TRUE
4	0	0	0	34.8200	0	FALSE

Period	u_t	K Tight	X Tight	C^{IM} Tight	Spread Tight	Binding Collateral
1	8	25.641	4.1848	8.7235	1.043	TRUE
2	20	32.039	6.2874	10.903	2.1305	TRUE
3	8	23.302	5.8256	24.669	0.18879	TRUE
4	0	0	0	34.953	0	FALSE

Period	u_t	K Det.	X Det.	C^{IM} Det.	Spread Det.	Binding Collateral
1	8	27.8125	4.0223	8.7235	1.6249	TRUE
2	20	34.1447	7.9045	13.2314	1.4573	TRUE
3	8	24.6471	6.1618	26.2914	0.1598	TRUE
4	0	0	0	36.9707	0	FALSE

Table III.5: Model dynamics of a path involving an extreme large realization of u_t with random and deterministic demands.

and rather less when u_t is high. This is because relative to the deterministic case, the extra risk from volatile shock intensity renders the risk-averse households willing to pay

	Utility_IM	Utility_A	Utility_B
Loose	11.9799	11.7964	12.5987
Tight	11.9915	11.7655	12.5679

Table III.6: Utility under two collateral policies.

more for hedging. Thus, all else equal, the price spread in the random case is larger than the deterministic one. So is the intermediaries' arbitrage profitability. When the collateral constraint is not binding, such as when u_t is extremely low, it is natural and also affordable for intermediaries with demand risk to take more positions. Likewise, when u_t is large, the collateral constraints endogenously become tighter, making it inhibitive for intermediaries to take extra positions.

Table III.6 presents agents' utility under two different collateral policies. It implies that intermediaries under tighter collateral policy enjoy higher welfare than their counterparts under looser collateral requirements. However, the opposite is true for households. Also, from the model dynamics in both normal time paths and those with rare events occurrence, intermediaries in the more liberal economy tend to take larger arbitrage positions at $t \in \{1, 2\}$. Accordingly, the price gaps between two markets are narrower than in the more restrictive economy. However, at $t = 3$, the relationship reverses. In fact, both market liquidity and capital investment in the looser environment are lower than those with tighter collateral policies in this period. Recall that in the four-period model, different collateral policies only affect constraints in period 1 and 2. In period three, the collateral constraints (III.14) in these two parallel economies are essentially identical, and the intermediaries' position sizes depend solely on their collateral input or their total income at the beginning of period three, given the constraint is binding. Hence, we can conclude that looser collateral constraints encourage intermediaries to provide more liquidity to the markets and allow them to better correct the mispricings. This in turn benefits households in segmented markets, as they can enjoy better risk sharing than their counterparts at an earlier stage. However, intermediaries in general benefit less from the arbitrage opportunities than their peers with tighter collateral constraints.

Since intermediaries under the looser collateral constraints can always replicate the arbitrage strategy and capital investment decisions of their peers in the other economy, one might wonder why they end up having less welfare? The rationale lies in the fact that intermediaries fail to internalize the price effects of their collective trading positions. The arbitrage profitability, measured by the price spreads, decreases with the aggregate liquidity that intermediaries provide to the markets. Hence, though intermediaries with looser collateral constraints are allowed to take on more arbitrage positions, they end up having less total profits due to this externality. Because of this, in period three when the income effects solely governs the intermediaries' arbitrage capacity, they can no longer maintain a larger position size or capital investment than their wealthier counterparts.

III.4.3 Risky Arbitrage with Unanticipated Demand Shocks

In the above experiment, we assume that agents all have perfect information of the underlying distribution of market demand, determined by $\{u_t\}$. However, in reality, most investors might not always be well informed. Instead, they are subject to unanticipated and extreme shocks. To examine the effects of information frictions on limits of arbitrage and aggregate activities, we conduct the following experiment in which agents do not have perfect information over the distribution of the shock intensity $\{u_t\}$. Given this incomplete information setting, we are further interested in studying the impact of collateral policies on the market stability in the event of unanticipated extreme shocks. Specifically, we analyze the sensitivity of key economic variables towards various unanticipated shocks under different collateral policies.

To achieve this goal, we implement a similar experiment with two parallel economies as in the first exercise. To model the responses towards unanticipated shocks, we assume that agents in both economies believe that u_t only takes the two values $u = 8$ and $u = 12$ with equal probability. The economy with loose collateral constraints imposes $\alpha = 50\%$ and the tighter one requires full loss coverage, i.e. $\alpha = 100\%$. Except for the incomplete information over $\{u_t\}$, agents have perfect knowledge about the other parameter values. We also inherit the same parameter settings of the two economies as in the previous exercise. We then simulate the model with the same set of $\{u_t\}$ paths and check the specific dynamics in equilibrium.

The full set of equilibrium dynamics is listed in section III.E in the appendix. Here for expositional convenience, we illustrate three representative sample paths in Table III.7. These paths stand for three distinct scenarios. The first one represents a path without unexpected shocks. The second features a path with an unanticipated large shock $u_2 = 20$, while the third involves an unexpected small size of $\{u_t\}$ with $u_2 = 0$.

In comparison with tables III.3, III.4 and III.5, we find that the aggregate capital investment as well as intermediaries' consumptions are lower, compared with the

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread Loose	Binding Collateral
1	8	25.659	4.8850	8.7235	0.8971	TRUE
2	12	27.013	5.9512	12.488	0.9508	TRUE
3	8	19.766	4.9415	20.482	0.2696	TRUE
4	0	0	0	29.649	0	FALSE
Period	u_t	K Tight	X Tight	C^{IM} Tight	Spread Tight	Binding Collateral
1	8	25.593	4.1843	8.7235	1.0317	TRUE
2	12	26.782	5.0571	12.532	1.0582	TRUE
3	8	19.711	4.9277	20.427	0.2693	TRUE
4	0	0	0	29.566	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread Loose	Binding Collateral
1	8	25.659	4.8850	8.7235	0.8971	TRUE
2	0	23.793	1.5319	13.629	0.3181	FALSE
3	8	18.045	4.5111	18.541	0.3010	TRUE
4	0	0	0	27.067	0	FALSE
Period	u_t	K Tight	X Tight	C^{IM} Tight	Spread Tight	Binding Collateral
1	8	25.593	4.1843	8.7235	1.0317	TRUE
2	0	23.866	1.5106	13.677	0.3169	FALSE
3	8	18.099	4.5247	18.603	0.2997	TRUE
4	0	0	0	27.148	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread Loose	Binding Collateral
1	8	25.659	4.8850	8.7235	0.8971	TRUE
2	20	31.953	7.2854	10.914	1.8243	TRUE
3	8	23.155	5.7887	24.486	0.1930	TRUE
4	0	0	0	34.732	0	FALSE
Period	u_t	K Tight	X Tight	C^{IM} Tight	Spread Tight	Binding Collateral
1	8	25.593	4.1843	8.7235	1.0317	TRUE
2	20	31.958	6.2678	10.874	2.1320	TRUE
3	8	23.247	5.8117	24.603	0.1900	TRUE
4	0	0	0	34.8700	0	FALSE

Table III.7: Three sample equilibrium paths with information friction under different collateral policies.

	Utility IM	Utility A	Utility B
Loose	11.9709	11.7970	12.5993
Tight	11.9831	11.7661	12.5685

Table III.8: Utility with information friction under two collateral policies.

counterparts with demand risk and perfect information as in the previous exercise. Also, except for $t = 1$ for the looser collateral case, the supply of market liquidity in the partially informed economies is in general less than in the fully informed ones. Interestingly, despite the smaller arbitrage volume, the price spreads in the less informed economies are yet narrower before $t = 3$, the time when all uncertainty unfolds. Table III.8 shows the agents' utility under two different collateral policies in the imperfect information settings. As we can see, intermediaries become worse off with information friction. In contrast, households in market A and B are better off.

Intuitively, due to the concave nature of the utility function, households are willing to pay more for the insurance against extremely large shocks, such as $u_t = 20$. Thus, all else equal, the price spreads are narrower when households are not aware of the possibility of extreme shocks. For intermediaries, this translates into the deterioration of their arbitrage profitability compared to the full information case. With narrower price spreads, intermediaries' overall marginal return of capital falls as its collateral value, measured by $\phi_t / \left(2\bar{\theta}\right)$, becomes less. Thus, it is natural for intermediaries to cut down their durable goods investment. Similarly, both the income effects and the weakened marginal profitability discourage intermediaries from maintaining the high level of market liquidity supply. Overall, intermediaries' welfare declines as their arbitrage opportunities get worse compared with the perfect information cases. However, as households underpay for the assets, the saved cost dominates the utility loss from less risk sharing. Due to the income effects, they are better off than in the full information cases.

Apart from the effects arising from the information friction, we also like to examine the net impact of different collateral policies on the economy's response towards various unanticipated shocks. Specifically, to separate the common influence of the imperfect information, we use the corresponding equilibrium paths in the full information settings as references. We then calculate the percentage changes to capture the shock sensitivity resulting from different collateral policies. In particular, we focus on comparing the relative effects at $t = 2$. This is because although unanticipated shocks might occur in both period

Period	u_t	K Loose	X Loose	Spread Loose	C^{IM} Loose	Liability Loose	Proceed Loose
1	8	-0.1850%	0.0500%	-1.1260%	0.0000%	—	-1.0760%
2	0	-0.2160%	-2.5210%	-1.3580%	-0.1380%	-1.3090%	-3.8450%
3	8	-0.1710%	-0.1710%	0.2100%	-0.1880%	-2.3190%	0.0400%

Period	u_t	K Tight	X Tight	Spread Tight	C^{IM} Tight	Liability Tight	Proceed Tight
1	8	0.0500%	-0.0130%	-1.0890%	0.0000%	—	-1.1010%
2	0	-0.2220%	-2.5320%	-1.3480%	-0.1450%	-1.3600%	-3.8460%
3	8	-0.1770%	-0.1770%	0.2200%	-0.1950%	-2.3210%	0.0400%

Period	u_t	K Loose	X Loose	Spread Loose	C^{IM} Loose	Liability Loose	Proceed Loose
1	8	-0.1850%	0.0500%	-1.1260%	0.0000%	—	-1.0760%
2	20	-0.2670%	-0.3260%	0.0800%	-0.2790%	0.1300%	-0.2490%
3	8	-0.2510%	-0.2510%	0.6800%	-0.2850%	0.3500%	0.4300%

Period	u_t	K Tight	X Tight	Spread Tight	C^{IM} Tight	Liability Tight	Proceed Tight
1	8	0.0500%	-0.0130%	-1.0890%	0.0000%	NA	-1.1010%
2	20	-0.2540%	-0.3120%	0.0700%	-0.2600%	0.0500%	-0.2460%
3	8	-0.2370%	-0.2370%	0.6500%	-0.2690%	0.3300%	0.4100%

Table III.9: The percentage changes under loose and tight collateral policies.

two and three, the collateral constraints in the two parallel economies become essentially identical at $t = 3$.

Table III.9 illustrates the percentage changes of equilibrium variables under different collateral policies when unanticipated shocks hit at $t = 2$, relative to the full information counterparts. To isolate the effects of externality, we further calculate the ratio of the loose policy cases over tight ones. For example, to compare the relative shock sensitivity on capital investment between two policies, we construct the ratio as

$$\hat{K} = \frac{K \text{ loose partial}/K \text{ loose full}}{K \text{ tight partial}/K \text{ tight full}}.$$

Table III.10 reports the relative sensitivity of equilibrium variables.

Both tables indicate that the economies under different collateral policies vary in their reactions towards unanticipated large and small shocks. In particular, in the presence of the large shock, i.e. $u_2 = 20$, intermediaries with the loose collateral constraints suffer from relatively higher liability and less immediate arbitrage proceeds. Consequently, relative to the levels under tougher collateral policy, their capital investment, liquidity supply and consumption are comparably lower. On the contrary, when the shock intensity is surprisingly small, i.e., $u_2 = 0$, intermediaries with looser collateral constraints tend to earn relatively more from the current arbitrage position x_2. Accordingly, they can afford to increase their consumptions, capital investment and liquidity supplies. In a nutshell, compared to their counterparts with tighter constraints, intermediaries under a looser collateral policy are more (less) vulnerable to unexpected large (small) shocks.

The distinct reactions towards different shocks arise from the fact that uninformed intermediaries in the more liberal economy tend to overinvest in their arbitrage positions at $t = 1$. This is due to both the looser constraints as well as the ignorance of potential risk from extreme shocks. Hence, when a surprisingly large shock hits at $t = 2$, the price spread widens and incurs heavier losses in their previous positions. Though the larger price gap also means higher arbitrage profitability, intermediaries' heavy losses render them less able to gather enough collateral from their own savings and to further take advantage of the arbitrage opportunities. Such income effect contributes to the relative reduction in their capital investment, consumption and liquidity supply.

In contrast, when the shock intensity is surprisingly low, i.e., $u_2 = 0$, the resulting price spread shrinks as the asset demands drop. This implies that the losses caused by intermediaries' previous aggressive arbitrage positions are much more limited. On

Period	u_t	$\hat{K}$	$\hat{X}$	$\hat{\phi}$	$\hat{C}^{\mathrm{IM}}$	Liability Rel. Ratio	Proceed Rel. Ratio
1	8	0.002%	0.063%	-0.037%	0.000%	NA	0.026%
2	0	0.006%	0.012%	-0.011%	0.007%	0.053%	0.001%
3	8	0.006%	0.006%	-0.010%	0.006%	0.002%	-0.004%

Period	u_t	$\hat{K}$	$\hat{X}$	$\hat{\phi}$	$\hat{C}^{\mathrm{IM}}$	Liability Rel. Ratio	Proceed Rel. Ratio
1	8	0.002%	0.063%	-0.037%	0.000%	NA	0.026%
2	20	-0.013%	-0.014%	0.010%	-0.019%	0.073%	-0.003%
3	8	-0.014%	-0.014%	0.031%	-0.016%	0.017%	0.017%

Table III.10: Relative sensitivity of looser collateral policy versus tighter one.

the other hand, the looser collateral policy emboldens the intermediaries to again take relatively larger arbitrage positions, thus earning more profits from providing liquidity. As a result, they are relatively less affected by the reduced arbitrage profitability. Put it differently, the extremely low demand shock, to some degree, exempts the intermediaries from their due "punishment" of their overinvestment. Since intermediaries under the looser collateral policy are more aggressive in trading at $t = 1$, their relative benefit from this kind of exemption is also larger.

III.5 Conclusion

We develop a model of collateral constrained arbitrage and use it to study the links between aggregate economic activities and the mispricings in the financial markets. Intermediaries exploit the price discrepancies between identical securities across segmented markets and meanwhile provide market liquidity through these market-making trades. In our deterministic version of the model, we illustrate the workings of the riskless arbitrage: while each of the legs of an arbitrage trade is risky, the risks offset each other across markets. With this, we examine the dynamics of durable goods investment, market liquidity and asset prices. We demonstrate the negative correlation between the aggregate capital investment level and the degree of mispricing in the financial markets.

In our risky arbitrage models, we first include the risk arising from volatile asset demands. We study the impact of the demand risk on welfare, market liquidity supply, asset prices as well as the aggregate output under different collateral policies. We show that with full awareness of the demand risk, intermediaries are able to gain higher arbitrage profits and invest more in capital while taking countercyclical trade volumes with respect to the arbitrage profitability. Also, thanks to the price externality effects, tighter collateral constraints allow intermediaries to better exploit the arbitrage opportunities and boost the aggregate investment and output in the real sectors. As an extension, we also introduce information friction to the risky arbitrage model. We investigate how different collateral policies might affect market stability and the economy's reaction towards various types of unanticipated shocks. We find that the economy with a more restrictive collateral requirement tends to be more robust towards large surprising demand shocks, while the one with looser collateral constraints is more resilient to unanticipated low-level shocks.

With the presence of information frictions, it is interesting to accommodate potential belief updating process of market participants in the future research. This will allow us to further explore how agents' belief, be it heterogeneous or homogeneous, affects the functioning of markets and the macroeconomic activities.

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III.A Proofs

III.A.1 Proof of Proposition III.1

Proof. The intermediaries Bellman equation is

$$V_t\left(S_t\right) = \max_{c^{\mathrm{IM}}_{t+1}, x_{t+1}, K_{t+1}} \left\{\rho \ln\left(c^{\mathrm{IM}}_{t+1}\right) + \rho V_{t+1}\left(S_{t+1}\right)\right\} \tag{III.15}$$

where

$$\begin{aligned} S_t &:= W_t - c^{\mathrm{IM}}_t = \left(Z_t + 1 - \delta\right) K_{t-1} - \phi_t x_{t-1} - c^{\mathrm{IM}}_t, \\ S_{t+1} &:= W_{t+1} - c^{\mathrm{IM}}_{t+1} = \left(Z_{t+1} + 1 - \delta\right) K_t - \phi_{t+1} x_t - c^{\mathrm{IM}}_{t+1} \\ &= \left(Z_{t+1} + 1 - \delta\right)\left(S_t + x_t \phi_t\right) - \phi_{t+1} x_t - c^{\mathrm{IM}}_{t+1}. \end{aligned}$$

As our conjecture is equivalent to

$$V_t\left(S_t\right) = C \ln\left(S_t\right) + D'_t,$$

substituting it to (III.15), we have

$$\begin{aligned} & C \ln\left(S_t\right) + D'_t \\ = & \max_{c^{\mathrm{IM}}_{t+1}, x_{t+1}, K_{t+1}} \left\{\rho \ln\left(c^{\mathrm{IM}}_{t+1}\right) + \rho C \ln\left[\left(Z_{t+1} + 1 - \delta\right)\left(S_t + x_t \phi_t\right) - \phi_{t+1} x_t - c^{\mathrm{IM}}_{t+1}\right]\right. \\ & \left. + \rho D'_{t+1}\right\}. \end{aligned}$$

The first-order condition with respect to consumption is

$$\frac{1}{c^{\mathrm{IM}}_{t+1}} - \frac{C}{\left(Z_{t+1} + 1 - \delta\right)\left(S_t + x_t \phi_t\right) - \phi_{t+1} x_t - c^{\mathrm{IM}}_{t+1}} = 0. \tag{III.16}$$

Thus,

$$c_{t+1}^{\text{IM}} = \frac{(Z_{t+1} + 1 - \delta)(S_t + x_t\phi_t) - x_t\phi_{t+1}}{C+1} = \frac{W_{t+1}}{C+1} = \frac{S_{t+1}}{C}.$$

With this, we can write the right-hand side of (III.15) as

$$\begin{aligned}&\max_{x_t} \{\rho(C+1)\ln[(Z_{t+1} + 1 - \delta)(S_t + x_t\phi_t) - x_t\phi_{t+1}] + \rho C \ln C \\ &-\rho(C+1)\ln(C+1) + \rho D'_{t+1}\}\end{aligned} \tag{III.17}$$

The maximization in (III.17) is subject to the financial constraint (III.2).

(a) When the collateral constraint is not binding, applying first order condition with respect to x_t yields

$$\phi_{t+1} - (Z_{t+1} + 1 - \delta)\phi_t = 0$$

That means, when the financial constraint is not binding, the intermediaries are indifferent between any position value of x_t.The above maximization becomes

$$\begin{aligned}C\ln(S_t) + D'_t &= \rho(C+1)\ln(S_t) + \rho(C+1)\ln(Z_{t+1} + 1 - \delta) + \rho C \ln C \\ &\quad -\rho(C+1)\ln(C+1) + \rho D'_{t+1}.\end{aligned}$$

Equating the coefficients in front of $\ln(S_t)$, we have:

$$\begin{aligned}&C = \frac{\rho}{1-\rho}, \\ &D'_t = \frac{\rho}{1-\rho}\ln(Z_{t+1} + 1 - \delta) + \frac{\rho^2}{1-\rho}\ln\rho + \rho\ln(1-\rho) + \rho D'_{t+1}.\end{aligned}$$

And the transversality condition is $\lim_{s\to\infty} \rho^s D'_{t+s} = 0$ determines D'_t.

(b) Otherwise, from the collateral constraint (III.2) we get

$$x_t = \frac{(1-\delta)K_t}{2\bar{\theta} + \phi_{t+1}} = \frac{(1-\delta)S_t}{2\bar{\theta} + \psi_{t+1} - (1-\delta)\phi_t},$$

where $\psi_{t+1} := \max\left\{p^B_{t+1} - p^A_{t+1}\right\}$. Substitute these to (III.17), the maximization is now

$$\begin{aligned}
C\ln\left(S_t\right) + D'_t &= \rho(1+C)\ln\left(S_t\right) \\
&\quad +\rho(1+C)\ln\left\{\left(Z_{t+1}+1-\delta\right) + \frac{\left[\left(Z_{t+1}+1-\delta\right)\phi_t - \phi_{t+1}\right](1-\delta)}{2\bar{\theta}+\psi_{t+1}-(1-\delta)\phi_t}\right\} \\
&\quad +\rho(C+1)\ln\left(Z_{t+1}+1-\delta\right) + \rho C\ln C - \rho(C+1)\ln(C+1) + \rho D'_{t+1}.
\end{aligned}$$

Again, equating the coefficients in front of $\ln(W_t)$ and the constant terms, we have

$$\begin{aligned}
C &= \frac{\rho}{1-\rho}, \\
D'_t &= \rho(1+C)\ln\left\{\left(Z_{t+1}+1-\delta\right) + \frac{\left[\left(Z_{t+1}+1-\delta\right)\phi_t - \phi_{t+1}\right](1-\delta)}{2\bar{\theta}+\psi_{t+1}-(1-\delta)\phi_t}\right\} \\
&\quad +\rho(C+1)\ln\left(Z_{t+1}+1-\delta\right) + \rho C\ln C - \rho(C+1)\ln(C+1) + \rho D'_{t+1}.
\end{aligned}$$

The transversality condition is $\lim_{s\to\infty}\rho^s D'_{t+s} = 0$ determines D'_t.

In sum, we have

$$\begin{aligned}
c^{\mathrm{IM}}_t &= \frac{W_t}{C+1} = (1-\rho)W_t, \\
S_t &= \rho W_t, \\
V\left(K_{t-1}, x_{t-1}\right) &= \frac{\rho}{1-\rho}\ln\left(W_t\right) + \frac{\rho}{1-\rho}\ln(\rho) + D'_t \\
&= \frac{\rho}{1-\rho}\ln\left\{\left(Z_t+1-\delta\right)K_{t-1} - x_{t-1}\phi_t\right\} + D_t,
\end{aligned}$$

where $D_t = \rho\ln(\rho)/(1-\rho) + D'_t$. □

III.A.2 Proof of Proposition III.2

Proof. The households' Bellman equation is

$$V_{i,t}\left(y^i_t\right) = \max_{c^i_{t+1}, y_{i,t+1}} \mathbb{E}_t\left[\beta\ln\left(c^i_{t+1}\right) + \beta V_{i,t+1}\left(y^i_{t+1}\right)\right].$$

substitute the budget constraint (III.4) in period $t+1$ and the conjecture (III.6) and apply iterated conditional expectation, we get

$$\begin{aligned}
\mathbb{E}\left[O\ln\left(y_t^i+N_t^i\right)+Q_t^i\right] \quad = \quad & \max_{c_{t+1}^i, y_{i,t+1}} \mathbb{E}_t\left[\beta\ln\left(y_t^i\left(\theta_{t+1}+P_{t+1}^i\right)+h+u_t^i\theta_{t+1}-P_{t+1}^i y_{t+1}^i\right)\right. \\
& \left.+\beta O\ln\left(y_{t+1}^i+N_{t+1}^i\right)+\beta Q_{t+1}^i\right] \qquad \text{(III.18)}
\end{aligned}$$

Apply the first-order condition with respect to the financial position y_{t+1} and we have

$$y_{t+1}^i=\frac{Oy_t\left(\theta_{t+1}+P_{t+1}^i\right)+Oh+Ou_t^i\theta_{t+1}-P_{t+1}^i N_{t+1}^i}{(1+O)P_{t+1}^i}.$$

Therefore, we can write the right-hand side of (III.18) as

$$\begin{aligned}
= \quad & \beta\mathbb{E}_t\left[\ln\left(y_t^i\left(\theta_{t+1}+P_{t+1}^i\right)+h+u_t^i\theta_{t+1}-\frac{Oy_t\left(\theta_{t+1}+P_{t+1}^i\right)+Oh+Ou_t^i\theta_{t+1}}{(1+A)}\right.\right. \\
& \left.\left.+\frac{P_{t+1}^i N_{t+1}^i}{1+O}\right)+O\ln\left(\frac{Oy_t\left(\theta_{t+1}+P_{t+1}^i\right)+Oh+Ou_t^i\theta_{t+1}-P_{t+1}^i N_{t+1}^i}{(1+O)P_{t+1}^i}+N_{t+1}^i\right)+Q_{t+1}^i\right] \\
= \quad & \beta\mathbb{E}_t\left[(1+O)\ln\left(y_t^i\left(\theta_{t+1}+P_{t+1}^i\right)+h+u_t^i\theta_{t+1}+N_{t+1}^i P_{t+1}^i\right)-(O+1)\ln(O+1)\right. \\
& \left.+O\ln O-O\ln\left(P_{t+1}^i\right)+Q_{t+1}^i\right] \\
= \quad & \beta\mathbb{E}_t\left[(1+O)\left(\ln\left(y_t^i+N_t^i\right)+\ln\left(\theta_{t+1}+P_{t+1}^i\right)+\ln(1+O)\right)+O\ln O-O\ln\left(P_{t+1}^i\right)+Q_{t+1}^i\right],
\end{aligned}$$

where we let

$$N_t^i=\frac{h+u_t^i\theta_{t+1}+N_{t+1}^i P_{t+1}^i}{\theta_{t+1}+P_{t+1}^i}.$$

Equating the coefficients in front of $\ln\left(y_t^i+N_t^i\right)$,

$$\beta(1+O)=O$$

thus we get $O=\beta/(1-\beta)$.

Equating others makes the Bellman equation hold for all values of y_t^i if

$$\begin{aligned}
O &= \frac{\beta}{1-\beta}, \\
N_t^i &= \frac{h + u_t^i \theta_{t+1} + N_{t+1}^i P_{t+1}^i}{\theta_{t+1} + P_{t+1}^i}, \\
Q_t^i &= \beta(1+O)\left(\ln\left(\theta_{t+1} + P_{t+1}^i\right) + \ln(1+O)\right) + \beta O \ln O - \beta O \ln\left(P_{t+1}^i\right) + \beta Q_{t+1}^i.
\end{aligned}$$

The transversality condition $\lim_{s\to\infty} \beta^s Q_{t+s}^i = 0$ determines Q_t^i. Similarly if we define $R_t^i \equiv \left(P_{t+1}^i + \theta_{t+1}\right)/P_t^i$, then the transversality condition $\lim_{s\to\infty} N_{t+s}^i / \prod_{m=t+1}^s R_m^i = 0$ also decides N_t^i.

□

III.A.3 Proof of Lemma 1

Proof of Lemma 1. We proof this lemma through backward induction. Suppose at $s = t+T$,

$$P_{t+T}^{\mathrm{A}}(\varepsilon) = -P_{t+T}^{\mathrm{B}}(-\varepsilon).$$

Define $C_s^i(\theta_s)$ to be the equilibrium consumption of HH in i at time s as a function of θ_s, for $i \in \{\mathrm{A}, \mathrm{B}\}$. Since $\{\theta_s\}$ follows a symmetric distribution around zero, then it must hold that

$$\frac{P_{t+T}^{\mathrm{A}}(\varepsilon)}{c_{t+T}^{\mathrm{A}}(\varepsilon)} = -\frac{P_{t+T}^{\mathrm{B}}(-\varepsilon)}{c_{t+T}^{\mathrm{B}}(-\varepsilon)}.$$

Thus,

$$\beta^T \mathbb{E}\left[\frac{p_{t+T}^{\mathrm{A}}}{c_{t+T}^{\mathrm{A}}}\right] = -\beta^T \mathbb{E}\left[\frac{p_{t+T}^{\mathrm{B}}}{c_{t+T}^{\mathrm{B}}}\right].$$

as $c_{t+T}^{\mathrm{A}}(\varepsilon) = c_{t+T}^{\mathrm{B}}(-\varepsilon)$, which follows from households' budget constraints.

At $s = T + t - 1$, from the first order condition of households, we have

$$\begin{aligned}
p_{t+T-1}^{\mathrm{A}} &= \beta c_{t+T-1}^{\mathrm{A}} \mathbb{E}\left[\frac{\theta_{t+T} + p_{t+T}^{\mathrm{A}}}{c_{t+T}^{\mathrm{A}}}\right], \\
p_{t+T-1}^{\mathrm{B}} &= \beta c_{t+T-1}^{\mathrm{B}} \mathbb{E}\left[\frac{\theta_{t+T} + p_{t+T}^{\mathrm{B}}}{c_{t+T}^{\mathrm{B}}}\right].
\end{aligned}$$

Substituting c^{A}_{t+T-1} and c^{B}_{t+T-1} with the households' budget constraints at $t+T-1$, it follows that

$$\begin{aligned} P^{\mathrm{A}}_{t+T-1}(\varepsilon) &= -P^{\mathrm{B}}_{t+T-1}(-\varepsilon), \\ c^{\mathrm{A}}_{t+T-1}(\varepsilon) &= c^{\mathrm{B}}_{t+T-1}(-\varepsilon). \end{aligned}$$

Likewise, one can derive

$$\begin{aligned} P^{\mathrm{A}}_{t}(\varepsilon) &= -P^{\mathrm{B}}_{t}(-\varepsilon), \\ c^{\mathrm{A}}_{t}(\varepsilon) &= c^{\mathrm{B}}_{t}(-\varepsilon). \end{aligned}$$

On the other hand, one can rewrite $P^{\mathrm{i}}_t(\varepsilon)$ as

$$\begin{aligned} P^{\mathrm{A}}_{t}(\varepsilon) &= c^{\mathrm{A}}_{t}(\varepsilon)\left(\sum_{j=1}^{T}\beta^j\mathbb{E}\left[\frac{\theta_{t+j}}{c^{\mathrm{A}}_{t+j}}\right]+\beta^T\mathbb{E}\left[\frac{p^{\mathrm{A}}_{t+T}}{c^{\mathrm{A}}_{t+T}}\right]\right) \\ &= -P^{\mathrm{B}}_{t}(-\varepsilon) \\ &= c^{\mathrm{B}}_{t}(-\varepsilon)\left(\sum_{j=1}^{T}\beta^j\mathbb{E}\left[\frac{\theta_{t+j}}{c^{\mathrm{B}}_{t+j}}\right]+\beta^T\mathbb{E}\left[\frac{p^{\mathrm{B}}_{t+T}}{c^{\mathrm{B}}_{t+T}}\right]\right). \end{aligned}$$

When $T \to \infty$, according to the TVC in market A and B,

$$\lim_{T\to\infty} -\beta^T\frac{p^{\mathrm{A}}_T}{c^{\mathrm{A}}_T}y^{\mathrm{A}}_T = 0,$$
$$\lim_{T\to\infty} \beta^T\frac{p^{\mathrm{B}}_T}{c^{\mathrm{B}}_T}y^{\mathrm{B}}_T = 0.$$

If the steady state prices $\lim_{T\to\infty} p^i_{t+T} \neq 0$, then it must hold $y^i_{t+T} \neq 0$ in equilibrium. Otherwise, some IM can make arbitrage profit by increasing liquidity providing. Thus, in this case,

$$\lim_{T\to\infty}\beta^T\mathbb{E}\left[\frac{p^{\mathrm{A}}_{t+T}}{c^{\mathrm{A}}_{t+T}}\right] = -\lim_{T\to\infty}\beta^T\mathbb{E}\left[\frac{p^{\mathrm{B}}_{t+T}}{c^{\mathrm{B}}_{t+T}}\right] = 0. \tag{III.19}$$

Else if $\lim_{T\to\infty} p^i_{t+T} = 0$, Equation (III.19) obviously holds as well.

Therefore, we have

$$
\begin{aligned}
P_t^{\mathrm{A}}(\varepsilon) &= c_t^{\mathrm{A}}(\varepsilon)\lim_{T\to\infty}\left(\sum_{j=1}^{T}\beta^j\mathbb{E}\left[\frac{\theta_{t+j}}{c_{t+j}^{\mathrm{A}}}\right]+\beta^T\mathbb{E}\left[\frac{P_{t+T}^{\mathrm{A}}}{c_{t+T}^{\mathrm{A}}}\right]\right)\\
&= c_t^{\mathrm{A}}(\varepsilon)\left(\sum_{j=1}^{\infty}\beta^j\mathbb{E}\left[\frac{\theta_{t+j}}{c_{t+j}^{\mathrm{A}}}\right]\right)\\
&= -P_t^{\mathrm{B}}(-\varepsilon)\\
&= c_t^{\mathrm{B}}(-\varepsilon)\left(\sum_{j=1}^{\infty}\beta^j\mathbb{E}\left[\frac{\theta_{t+j}}{c_{t+j}^{\mathrm{B}}}\right]\right).
\end{aligned}
$$

□

III.A.4 Proof of Proposition III.3

Proof. From Lemma 1, we know that

$$P_t^{\mathrm{A}}(\varepsilon) = -P_t^{\mathrm{B}}(-\varepsilon).$$

Also, from households' budget constraints,

$$
\begin{aligned}
c_t^{\mathrm{A}}(\varepsilon) &= -P_t^{\mathrm{A}}(\varepsilon)\left(y_t^{\mathrm{A}}-y_{t-1}^{\mathrm{A}}\right)+h_t+\left(u+y_{t-1}^{\mathrm{A}}\right)\varepsilon\\
&= P_t^{\mathrm{A}}(\varepsilon)\left(x_t-x_{t-1}\right)+h_t+\left(u-x_{t-1}\right)\varepsilon,
\end{aligned}
$$

Similarly,

$$c_t^{\mathrm{B}}(\varepsilon) = -P_t^{\mathrm{B}}(\varepsilon)\left(x_t-x_{t-1}\right)+h_t-\left(u-x_{t-1}\right)\varepsilon.$$

Thus, it is obvious that

$$
\begin{aligned}
c_t^{\mathrm{A}}(\varepsilon) &= c_t^{\mathrm{B}}(-\varepsilon),\\
\mathbb{E}\left[\frac{\theta_{t+s}+p_{t+s}^{\mathrm{A}}}{c_{t+s}^{\mathrm{A}}}\right] &= -\mathbb{E}\left[\frac{\theta_{t+s}+p_{t+s}^{\mathrm{B}}}{c_{t+s}^{\mathrm{B}}}\right],\quad \forall s\in\{1,2,\dots\}.
\end{aligned}
$$

From households' first order conditions, it follows that

$$\begin{aligned}
\phi_t &\equiv p_t^{\mathrm{B}} - p_t^{\mathrm{A}} \\
&= \beta c_t^{\mathrm{B}} \mathbb{E}\left[\frac{\theta_{t+1} + p_{t+1}^{\mathrm{B}}}{c_{t+1}^{\mathrm{B}}}\right] - \beta c_t^{\mathrm{A}} \mathbb{E}\left[\frac{\theta_{t+1} + p_{t+1}^{\mathrm{A}}}{c_{t+1}^{\mathrm{A}}}\right] \\
&= \beta\left(c_t^{\mathrm{A}} + c_t^{\mathrm{B}}\right) \mathbb{E}\left[\frac{\theta_{t+1} + p_{t+1}^{\mathrm{B}}}{c_{t+1}^{\mathrm{B}}}\right].
\end{aligned}$$

Thus,

$$p_t^{\mathrm{B}} = \beta c_t^{\mathrm{B}} \mathbb{E}\left[\frac{\theta_{t+1} + p_{t+1}^{\mathrm{B}}}{c_{t+1}^{\mathrm{B}}}\right] = \frac{c_t^{\mathrm{B}}}{c_t^{\mathrm{A}} + c_t^{\mathrm{B}}} \phi_t.$$

Likewise,

$$p_t^{\mathrm{A}} = -\frac{c_t^{\mathrm{A}}}{c_t^{\mathrm{A}} + c_t^{\mathrm{B}}} \phi_t.$$

Substituting c_t^{i} with households' budget constraints in i, $i \in \{\mathrm{A}, \mathrm{B}\}$, one can obtain the following after rearranging

$$p_t^{\mathrm{B}} = \frac{h - (u - x_{t-1})\,\theta_t}{2h} \phi_t, \qquad p_t^{\mathrm{A}} = -\frac{h + (u - x_{t-1})\,\theta_t}{2h} \phi_t.$$

On the other hand, if we continue decompose ϕ_t,

$$\begin{aligned}
\phi_t &= \beta c_t^{\mathrm{B}} \mathbb{E}\left[\frac{\theta_{t+1} + p_{t+1}^{\mathrm{B}}}{c_{t+1}^{\mathrm{B}}}\right] - \beta c_t^{\mathrm{A}} \mathbb{E}\left[\frac{\theta_{t+1} + p_{t+1}^{\mathrm{A}}}{c_{t+1}^{\mathrm{A}}}\right] \\
&= \beta\left(-p_t^{\mathrm{B}}\left(x_t - x_{t-1}\right) + h - \left(u - x_{t-1}\right)\theta_t\right) \mathbb{E}\left[\frac{\theta_{t+1} + p_{t+1}^{\mathrm{B}}}{c_{t+1}^{\mathrm{B}}}\right] \\
&\quad -\beta\left(p_t^{\mathrm{A}}\left(x_t - x_{t-1}\right) + h + \left(u - x_{t-1}\right)\theta_t\right) \mathbb{E}\left[\frac{\theta_{t+1} + p_{t+1}^{\mathrm{A}}}{c_{t+1}^{\mathrm{A}}}\right] \\
&= \beta\left(-\phi_t\left(x_t - x_{t-1}\right) + 2h\right) \mathbb{E}\left[\frac{\theta_{t+1} + p_{t+1}^{\mathrm{B}}}{c_{t+1}^{\mathrm{B}}}\right].
\end{aligned}$$

After rearranging and repeatedly substituting with households' first order condition, we can get

$$\phi_t \equiv p_t^{\mathrm{B}} - p_t^{\mathrm{A}} = \frac{2h}{M_t + (x_t - x_{t-1})}.$$

where

$$1/M_t := \sum_{j=1}^{\infty} \beta^j \mathbb{E}_t\left[\frac{\theta_{t+j}}{c_{t+j}^{\mathrm{B}}}\right],$$

which is independent of the realization of θ_t.

As intermediaries take a net zero position in the financial markets, when u_t is constant, their optimization problems are deterministic. Accordingly, x_t, x_{t-1}, K_{t-1} are all independent of the realization of θ_t in equilibrium. Hence the price difference ϕ_t doesn't depend on any particular θ_t realization.

$\square$

III.A.5 Proof of Proposition III.4

Proof. Substituting c_t^{IM} from Equation (III.5) into intermediaries' budget constraint and solve for K_t, we get

$$K_t = \rho \left(Z_t + 1 - \delta\right) K_{t-1} + x_t \phi_t.$$

If the collateral constraint in t is slack, i.e., $\phi_t = 0$, $x_t = u$, then K_t must satisfy

$$(1-\delta)K_t \geq u\left(\phi_{t+1} + 2\bar{\theta}\right) \geq 2u\bar{\theta}.$$

That is,

$$K_t = \rho \left(Z_t + 1 - \delta\right) K_{t-1} \geq \frac{2u\bar{\theta}}{1-\delta}.$$

Thus, once

$$K_{t-1} \geq \frac{2u\bar{\theta}}{\rho(1-\delta)\left(Z_t + 1 - \delta\right)},$$

$G_t = Z_t + 1 - \delta$.

Otherwise if

$$K_{t-1} < \frac{2u\bar{\theta}}{\rho(1-\delta)\left(Z_t + 1 - \delta\right)}$$

and the collateral constraint in t is binding, i.e., $\phi_t > 0$ and $x_t < u$, then combining collateral constraint (III.3), we further obtain

$$K_t = \rho \left(Z_t + 1 - \delta\right) K_{t-1} + x_t \phi_t = \rho \left(Z_t + 1 - \delta\right) K_{t-1} + \frac{(1-\delta)K_t\phi_t}{\phi_{t+1} + 2\bar{\theta}},$$

Rearranging, we get

$$K_t = \frac{\rho \left(Z_t + 1 - \delta\right) K_{t-1}}{1 - (1-\delta)K_t\phi_t / \left(\phi_{t+1} + 2\bar{\theta}\right)}.$$

Thus,

$$G_t = \frac{(Z_t + 1 - \delta)}{1 - (1-\delta)K_t\phi_t / \left(\phi_{t+1} + 2\bar{\theta}\right)}.$$

From intermediaries' first order condition,

$$\left(1 - \frac{(1-\delta)K_t\phi_t}{\phi_{t+1} + 2\bar{\theta}}\right)^{-1} > 1.$$

Thus, $G_t > Z_t + 1 - \delta$.

$\square$

III.A.6 Proof of Proposition III.6

Proof. From Proposition III.4, the dynamics of K_t in the four cases implies the following.

- if $\rho(Z + 1 - \delta) > 1$, then Equation (III.10) indicates that $\rho G_t > 1$, K_t will keep growing to $K = \infty$. Intermediaries' collateral constraints will eventually become slack, i.e., x_t grows to $x^* = u$ and ϕ_t decreases to $\phi^* = 0$.

- if $\rho(Z + 1 - \delta) = 1$, then Equation (III.10) indicates that if $K_0 \geq 2u\bar{\theta}/(1-\delta)$, then collateral constraints are always slack, $\rho G_t = 1$ for all t. Thus, K_t will stay at $K^* = K_0$, with $x_t = x^* = u$ and $\phi_t = \phi^* = 0$ for all t. Else, if $K_0 < 2u\bar{\theta}/(1-\delta)$, initially the collateral constraints are binding, i.e., $\rho G_t > \rho(Z+1-\delta) = 1$. Hence, K_t will keep growing towards $K^* = 2u\bar{\theta}/(1-\delta)$ until the collateral constraints become slack. x_t will grow to $x^* = u$ and ϕ_t decreases to $\phi^* = 0$.

- If $\rho(Z + 1 - \delta) < 1$ and $\rho\bar{G} < 1$, where $\bar{G}$ is the maximal of G_t for all t. Then Equation (III.10) indicates that $\rho G_t < 1$, K_t will eventually decreases towards $K^* = 0$, together with x_t diminishing to $x^* = 0$. When $x^* = 0$, households enjoy zero risk sharing. Thus, from the households' first order condition, in such a steady state with $x^* = 0$,

$$
\begin{aligned}
\phi_t^* &:= P_t^{\mathrm{B}*} - P_t^{\mathrm{A}*} \\
&= \beta c_t^{\mathrm{B}*} \mathbb{E}\left[\frac{P_{t+1}^{\mathrm{B}*} + \theta_{t+1}}{c_{t+1}^{\mathrm{B}*}}\right] - \beta c_t^{\mathrm{A}*} \mathbb{E}\left[\frac{P_{t+1}^{\mathrm{A}*} + \theta_{t+1}}{c_{t+1}^{\mathrm{A}*}}\right] \\
&= \beta\left(c_t^{\mathrm{B}*} + c_t^{\mathrm{A}*}\right) \mathbb{E}\left[\frac{P_{t+1}^{\mathrm{B}*} + \theta_{t+1}}{c_{t+1}^{\mathrm{B}*}}\right] \\
&= 2\beta h\left(\frac{\phi_{t+1}}{c_{t+1}^{\mathrm{A}*} + c_{t+1}^{\mathrm{B}*}} + \mathbb{E}\left[\frac{\theta_{t+1}}{c_{t+1}^{\mathrm{B}*}}\right]\right) \\
&= 2\beta h\left(\frac{\phi_{t+1}^*}{2h} + \mathbb{E}\left[\frac{\theta_{t+1}}{h - u\theta_{t+1}}\right]\right).
\end{aligned}
$$

Rearranging, we get

$$
\phi^* = 2h\frac{\beta}{1-\beta}\mathbb{E}_t\left[\frac{\theta_{t+1}}{h - u\theta_{t+1}}\right].
$$

- Otherwise, Equation (III.10) implies that K_t remains constant when ρG_t is equal to one as its steady state value, i.e., $\rho G^* = 1$. In the steady state, intermediaries' collateral constraints are binding. Solving $\rho G^* = 1$, we get

$$
\phi^* = 2\frac{1 - \rho(Z + 1 - \delta)}{\rho Z - \delta}\bar{\theta}.
$$

On the other hand, from households' first order condition, ϕ^* also has to satisfy

$$
\begin{aligned}
\phi^* &= P_t^{\mathrm{B}*} - P_t^{\mathrm{A}*} \\
&= \beta c_t^{\mathrm{B}*} \mathbb{E}\left[\frac{P_{t+1}^{\mathrm{B}*} + \theta_{t+1}}{c_{t+1}^{\mathrm{B}*}}\right] - \beta c_t^{\mathrm{A}*} \mathbb{E}\left[\frac{P_{t+1}^{\mathrm{A}*} + \theta_{t+1}}{c_{t+1}^{\mathrm{A}*}}\right] \\
&= \beta\left(c_t^{\mathrm{B}*} + c_t^{\mathrm{A}*}\right) \mathbb{E}\left[\frac{P_{t+1}^{\mathrm{B}*} + \theta_{t+1}}{c_{t+1}^{\mathrm{B}*}}\right] \\
&= 2\beta h\left(\frac{\phi_{t+1}}{c_{t+1}^{\mathrm{A}*} + c_{t+1}^{\mathrm{B}*}} + \mathbb{E}\left[\frac{\theta_{t+1}}{c_{t+1}^{\mathrm{B}*}}\right]\right) \\
&= 2\beta h\left(\frac{\phi^*}{2h} + \mathbb{E}\left[\frac{\theta_{t+1}}{h - (u - x^*)\,\theta_{t+1}}\right]\right).
\end{aligned}
$$

Thus, we can solve for the unique steady state x^* that satisfy the above. The binding collateral constraints indicates that $K^* = x^*\left(\phi^* + 2\bar{\theta}\right)/(1-\delta)$.

□

III.A.7 Proof of Proposition III.5

Proof. Suppose that in equilibrium (1) the positions in market A and B are opposite with the same absolute size; (2) given the price gap, the intermediaries and the households' optimization problems are solved in Section III.2.6; (3) the price gap ϕ_t are given either by solving the agents' optimization problem backwards with the terminal condition $\phi^* = 0$ for the slack case or

$$\phi^* = 2\frac{1 - \rho(Z + 1 - \delta)}{\rho Z - \delta}\bar{\theta}$$

for the binding case in Section III.3.2 through market clearing conditions. Thus the equilibrium exist.

□

III.B The Infinite Horizon Model with Risky Arbitrage

III.B.1 Recursive Formulation

For the general case, it is remarkably difficult to analytically solve for the prices and capital accumulation in the sequential trading economy, especially we do not know the mappings from the historical path of productivity shocks and shock intensity. Therefore, we try to avoid the difficulty by restating the equilibrium through recursive formulation. Based on the recursive structure, we consider applying numerical algorithms to find equilibrium prices and asset allocations.

Intermediaries

Denote the intermediaries' current position in financial market A as $x = x_{\mathrm{A},t}$. Therefore, their position in market B is $-x$. Similarly, the A-households' position in equilibrium will be $y_{\mathrm{A}} = -x$ and the B-households' position is $y_{\mathrm{B}} = x$. Also in the following formulation, we denote the intermediaries' capital input and position in market A for the previous period by $K_{\mathrm{P},-}$ and x_-. Similarly, the shock intensity of the previous and current periods in market A are u_- and u respectively.

To form a recursive expression for the intermediaries, we use the last period's and current period's shock intensities u_- and u, the current period's productivity shock θ, the intermediaries' physical capital P holding at the beginning of the period $K_{\mathrm{P},-}$ within each market, and the previous period's position in the financial asset in market A $x_- = x_{\mathrm{A},-}$ as state variables.

Also we denote the next period production shock and asset prices as θ_+ and $P_{i,+}$ for $i \in \{\mathrm{A}, \mathrm{B}\}$.

The recursive formulation for the intermediaries can be expressed as

$$V^{\mathrm{IM}}\left(K_{\mathrm{P},-}, x_-, u_-, \theta\right) = \max_{K_{\mathrm{P}}, x, c^{\mathrm{IM}}} \left\{\ln\left(c^{\mathrm{IM}}\right) + \rho \mathbb{E}\left[V^{\mathrm{IM}}(K_{\mathrm{P}}, x, u, \theta_+)\right]\right\}$$

subject to the budget constraints

$$\begin{aligned} c^{\mathrm{IM}} &= \left(P_{\mathrm{A}} - P_{\mathrm{B}}\right)\left(x_- - x\right) + a(1-\gamma)K_{\mathrm{P},-}^{\alpha}L^{\gamma} + (1-\delta)K_{\mathrm{P},-} - K_{\mathrm{P}}, && \text{(III.20)} \\ c_+^{\mathrm{IM}} &= \left(P_{\mathrm{A},+} - P_{\mathrm{B},+}\right)\left(x - x_+\right) + a(1-\gamma)K_{\mathrm{P}}^{\alpha}L^{\gamma} + (1-\delta)K_{\mathrm{P}} - K_{\mathrm{P},+}. && \text{(III.21)} \end{aligned}$$

and subject to the collateral constraints

$$\begin{aligned} \text{(type I)} \quad 0 \ &\leq \ \min\{\min_{P_{\mathrm{A},+}}\left\{x\left(P_{\mathrm{A},+} - P_{\mathrm{A}}\right)\right\}, 0\} + \min\{\min_{P_{\mathrm{B},+}}\left\{(-x)\left(P_{\mathrm{B},+} - P_{\mathrm{B}}\right)\right\}, 0\} \\ &\quad + a(1-\gamma)K_{\mathrm{P}}^{\alpha}L^{\gamma} + (1-\delta)K_{\mathrm{P}}, \\ \text{(type II)} \quad 0 \ &\leq \ \min\{\min_{P_{\mathrm{A},+}}\left\{x\left(P_{\mathrm{A},+} - P_{\mathrm{A}}\right)\right\}, 0\} + \min\{\min_{P_{\mathrm{B},+}}\left\{(-x)\left(P_{\mathrm{B},+} - P_{\mathrm{B}}\right)\right\}, 0\} + K_{\mathrm{P}}. \end{aligned}$$

Households

For households, the recursive form is easier as they are not subject to any collateral constraints. Their maximization problem can also be expressed in a recursive form with state variables $K_{\mathrm{P},-}$, x_-, u_-, u, and θ. This is because of the market clearing condition and the symmetry between two markets. The A-households' position in the financial market is $y = -x$, and the B-households' position is therefore equal to x in equilibrium. Thus, x can serve as state variable in the households' recursive formulation as well.

The A-households' optimization problem in a recursive form is given by

$$V^{\mathrm{A}}\left(K_{\mathrm{P},-}, x_-, u_-, \theta\right) = \max_{y_{\mathrm{A}}, c^{\mathrm{A}}} \left\{\ln\left(c^{\mathrm{A}}\right) + \beta \mathbb{E}\left[V^{\mathrm{A}}\left(K_{\mathrm{P}}, x, u, \theta_+\right)\right]\right\}$$

$$
\begin{aligned}
c^{\mathrm{A}} &= -P_{\mathrm{A}}\left(x_{-}+y_{\mathrm{A}}\right)+\frac{1}{2} a \gamma K_{\mathrm{P},-}^{\alpha} L^{\gamma}+b K_{\mathrm{F}}+\left(u_{-}-x_{-}\right) \theta, && \text{(III.22)}\\
c_{+}^{\mathrm{A}} &= -P_{\mathrm{A},+}\left(-y_{\mathrm{A}}+y_{\mathrm{A},+}\right)+\frac{1}{2} a \gamma K_{\mathrm{P}}^{\alpha} L^{\gamma}+b K_{\mathrm{F}}+\left(u+y_{\mathrm{A}}\right) \theta_{+}. && \text{(III.23)}
\end{aligned}
$$

The corresponding B-households' optimization problem is given by

$$
V^{\mathrm{B}}\left(K_{\mathrm{P},-}, x_{-}, u_{-}, \theta\right)=\max_{y_{\mathrm{B}}, c^{\mathrm{B}}}\left\{\ln \left(c^{\mathrm{B}}\right)+\beta \mathbb{E}\left[V^{\mathrm{B}}\left(K_{\mathrm{P}}, x, u, \theta_{+}\right)\right]\right\}
$$

subject to the budget constraints

$$
\begin{aligned}
c^{\mathrm{B}} &= P_{\mathrm{B}}\left(x_{-}-y_{\mathrm{B}}\right)+\frac{1}{2} a \gamma K_{\mathrm{P},-}^{\alpha} L^{\gamma}+b K_{\mathrm{F}}-\left(u_{-}-x_{-}\right) \theta, && \text{(III.24)}\\
c_{+}^{\mathrm{B}} &= P_{\mathrm{B},+}\left(y_{\mathrm{B}}-y_{\mathrm{B},+}\right)+\frac{1}{2} a \gamma K_{\mathrm{P}}^{\alpha} L^{\gamma}+b K_{\mathrm{F}}-\left(u-y_{\mathrm{B}}\right) \theta_{+}. && \text{(III.25)}
\end{aligned}
$$

The market clearing condition implies that

$$
\begin{aligned}
y_{\mathrm{A}} &= -x, && \text{(III.26)}\\
y_{\mathrm{B}} &= x, && \text{(III.27)}\\
y_{\mathrm{A},+} &= -x_{+}, && \text{(III.28)}\\
y_{\mathrm{B},+} &= x_{+}. && \text{(III.29)}
\end{aligned}
$$

III.B.2 Garcia and Zangwill (1981) Trick

In order to transform the inequality into an equality, we apply a change of variables following Garcia and Zangwill (1981). Then the transformed equality for the intermediaries' optimality problem is given by

$$
V^{\mathrm{IM}}\left(K_{\mathrm{P},-}, x_{-}, u_{-}, \theta\right)=\max_{K_{\mathrm{P}}, x, c^{\mathrm{IM}}}\left\{\ln \left(c^{\mathrm{IM}}\right)+\rho \mathbb{E}\left[V^{\mathrm{IM}}(K_{\mathrm{P}}, x, u, \theta_{+}\right]\right\}
$$

subject to

$$
c^{\mathrm{IM}}=\left(P_{\mathrm{A}}-P_{\mathrm{B}}\right)\left(x_{-}-x\right)+a(1-\gamma) K_{\mathrm{P},-}^{\alpha} L^{\gamma}+(1-\delta) K_{\mathrm{P},-}-K_{\mathrm{P}}
$$

and

$$
\begin{aligned}
\text{(type I)} \quad 0 \; &= \lambda_{1,P_+}(-) - \{\min\left[x(P_{\mathrm{A},+} - P_{\mathrm{A}})\right] + \min\left[(-x)(P_{\mathrm{B},+} - P_{\mathrm{B}})\right] \\
&\quad + a(1-\gamma)K_{\mathrm{P}}^{\alpha}L^{\gamma} + (1-\delta)K_{\mathrm{P}}\}\,, \qquad \text{(III.30a)} \\
\text{(type II)} \quad 0 \; &= \lambda_{2,P_+}(-) - \{\min\left[x(P_{\mathrm{A},+} - P_{\mathrm{A}})\right] + \min\left[(-x)(P_{\mathrm{B},+} - P_{\mathrm{B}})\right] + K_{\mathrm{P}}\} \qquad \text{(III.30b)}
\end{aligned}
$$

for all $u_+ \in \mathcal{U}$. Here,

$$
\begin{aligned}
\lambda_{1,P_+}(-) &= \left(\max\left\{-\lambda_{1P_+}, 0\right\}\right)^2, \\
\lambda_{2,P_+}(-) &= \left(\max\left\{-\lambda_{2,P_+}, 0\right\}\right)^2
\end{aligned}
$$

are the inequality multipliers in Garcia and Zangwill (1981).

III.B.3 First Order Conditions

The first order conditions for the intermediaries with two different friction types are given by

$$
\begin{aligned}
\frac{1}{c^{\mathrm{IM}}} \; &= \rho\mathbb{E}\left[\frac{1}{c_+^{\mathrm{IM}}}\left(a\alpha(1-\gamma)K_{\mathrm{P}}^{\alpha-1}L^{\gamma} + (1-\delta)\right)\right] \\
&\quad + \lambda_{1,P_+}(+)\left(a\alpha(1-\gamma)K_{\mathrm{P}}^{\alpha-1}L^{\gamma} + (1-\delta)\right), \qquad \text{(III.31)} \\
\frac{1}{c^{\mathrm{IM}}}\left(P_{\mathrm{A}} - P_{\mathrm{B}}\right) \; &= \rho\mathbb{E}\left[\frac{1}{c_+^{\mathrm{IM}}}\left(P_{\mathrm{A},+} - P_{\mathrm{B},+}\right)\right] \\
&\quad + \lambda_{1,P_+}(+)\{\mathbb{1}_x \min(P_{\mathrm{A},+}) + (1-\mathbb{1}_x)\max(P_{\mathrm{A},+}) - P_{\mathrm{A}} \\
&\quad - \left[\mathbb{1}_x \max(P_{\mathrm{B},+}) + (1-\mathbb{1}_x)\min(P_{\mathrm{B},+})\right] + P_B\}. \qquad \text{(III.32)}
\end{aligned}
$$

in case of type I frictions and

$$\begin{aligned}
\frac{1}{c^{\mathrm{IM}}} &= \rho\mathbb{E}\left[\frac{1}{c_{+}^{\mathrm{IM}}}\left(a\alpha(1-\gamma)K_{\mathrm{P}}^{\alpha-1}L^{\gamma}+(1-\delta)\right)\right]+\lambda_{2,P_{+}}(+), && \text{(III.33)}\\
\frac{1}{c^{\mathrm{IM}}}(P_A-P_B) &= \rho\mathbb{E}\left[\frac{1}{c_{+}^{\mathrm{IM}}}\left(P_{\mathrm{A},+}-P_{\mathrm{B},+}\right)\right]\\
&\quad+\lambda_{2,P_{+}}(+)\{\mathbb{1}_x\min(P_{\mathrm{A},+})+(1-\mathbb{1}_x)\max(P_{\mathrm{A},+})-P_{\mathrm{A}}\\
&\quad-[\mathbb{1}_x\max(P_{\mathrm{B},+})+(1-\mathbb{1}_x)\min(P_{\mathrm{B},+})]+P_B\} && \text{(III.34)}
\end{aligned}$$

in case of type II frictions, where $\mathbb{1}_x$ is the sign indicator of x. Here,

$$\begin{aligned}
\lambda_{1,P_{+}}(+) &= \left(\max\left\{\lambda_{1,P_{+}},0\right\}\right)^2,\\
\lambda_{2,P_{+}}(+) &= \left(\max\left\{\lambda_{2,P_{+}},0\right\}\right)^2
\end{aligned}$$

are the inequality multipliers in Garcia and Zangwill (1981).

The corresponding first order conditions for the households read

$$\begin{aligned}
\frac{P_{\mathrm{A}}}{c^{\mathrm{A}}} &= \beta\mathbb{E}\left[\frac{P_{\mathrm{A},+}+\theta_{+}}{c_{+}^{\mathrm{A}}}\right], && \text{(III.36)}\\
\frac{P_{\mathrm{B}}}{c^{\mathrm{B}}} &= \beta\mathbb{E}\left[\frac{P_{\mathrm{B},+}+\theta_{+}}{c_{+}^{\mathrm{B}}}\right]. && \text{(III.37)}
\end{aligned}$$

III.C Numerical Implementation

To derive a quantitative characterization of the equilibrium, we apply a similar approximation algorithm to approximate the equilibrium asset allocations and prices as in Kübler and Schmedders (2003).

The six budget constraint equations (III.20) through (III.25), the four market clearing conditions (III.26) through (III.29), together with one collateral constraint (III.30a)/(III.30b) and four first order condition equations (III.31), (III.33), (III.32) and (III.34) constitute a non-linear system of equations in 19 unknowns. It consists of

- four price variables P_{A}, $P_{\mathrm{A},+}$, P_{B}, $P_{\mathrm{B},+}$,
- six consumption variables c^{IM}, c_{+}^{IM}, c^{A}, c_{+}^{A}, c^{B}, c_{+}^{B},

- asset position variables x, x_+, y_{A}, $y_{\mathrm{A},+}$, y_{B}, $y_{\mathrm{B},+}$,
- two capital accumulation levels K_{P}, $K_{\mathrm{P},+}$,
- and one multiplier $\lambda_{1,P_+}/\lambda_{2,P_+}$.

Denote the above equations as $\boldsymbol{F}(x_-, K_{\mathrm{P},-}, u, u_-, \theta) = \mathbf{0}_{15}$, where $\mathbf{0}_n$ is an n-dimensional column vector of zeros.

We compute an approximate policy function via an iterative algorithm. In particular, we assume that the values of the prices, the endogenous intermediaries' financial asset allocation and capital accumulation in the next period, $P_{\mathrm{A},+}$, $P_{\mathrm{B},+}$, x_+ and $K_{\mathrm{P},+}$, are functions of the intermediaries' current capital accumulation $K_{\mathrm{P},-}$ and financial asset holdings x_-, which we denote as

$$\boldsymbol{\zeta} : \underbrace{[-u, u]}_{x_-} \times \underbrace{\mathbb{R}_+}_{K_{\mathrm{P},-}} \times \mathcal{U} \times \mathcal{U} \times \mathcal{S} \to \underbrace{[-u, u]}_{x_+} \times \underbrace{\mathbb{R}}_{P_{\mathrm{A},+}} \times \underbrace{\mathbb{R}}_{P_{\mathrm{B},+}} \times \underbrace{\mathbb{R}_+}_{K_{\mathrm{P},+}} .$$

By approximating those variables as functions of current state, we manage to transform $\boldsymbol{F}(x_-, K_{\mathrm{P},-}, u, u_-, \theta) = 0$ into a well-defined system of equations. As a starting point, we choose a continuous function $\boldsymbol{\zeta}^0$ to serve as an initial guess to approximate the next period's prices, asset allocations and capital accumulation levels. During each iteration of the algorithm, given the approximated next period prices and asset holdings $\boldsymbol{\zeta}^n$, we solve the well-defined system to obtain the equilibrium prices and holdings in the current period. We then move one period back, and update the approximation of $\boldsymbol{\zeta}^{n+1}$ by mapping the current period values to the state variables. We define the convergence of the iteration by a predetermined criterion. For some predetermined $\epsilon > 0$,

$$\sup_{\substack{x_-, K_{\mathrm{P},-}, \\ u, u_-, \theta}} \|\boldsymbol{\zeta}^{n+1} - \boldsymbol{\zeta}^n\| \leq \epsilon.$$

The algorithm terminates once $\boldsymbol{\zeta}^{n+1}$ reaches convergence and we accept $\boldsymbol{\zeta}^* = \boldsymbol{\zeta}^{n+1}$ as approximated price and policy functions for the next period.

At the end, we will compute the maximum relative errors in Euler equations after substituting the approximated value into the first order conditions to examine the quality of the approximation. For errors that exceeds our preset criterion, we will restart the above iteration with a lower ϵ as convergence threshold.

III.C.1 Implementation Procedure

Specifically, we construct a piecewise linear spline with coefficient ξ^0 to obtain approximation ζ^0 as an initial set up. In each iteration given ζ^n, we solve the above system of nonlinear equations (III.20) through (III.37) and thus obtain the value of current period prices and control variables. Then we interpolate them against the state variables and get an updated ζ^{n+1} by updating the coefficient vectors ξ^{n+1} and. Repeat this procedure until the convergence of ξ^{n+1}.

III.C.2 Algorithm

We applied the following time iteration linear collocation algorithm similar to Kübler and Schmedders (2003) and Judd (1998).

- Step 0: select an error tolerance ϵ for the stopping criterion $\sup \|\zeta^{n+1} - \zeta^n\| \leq \epsilon$, a finite grid composed of $K_{\mathrm{P},-} \in [0, K]$ and $x_- \in [-u, u]$ for each combination of (u_-, u, θ) and the piecewise linear coefficients ξ^0 for a starting point ζ^0.

- Step 1: Given the piecewise linear coefficients ξ^n, or the approximation ζ^n, solve the system of nonlinear equations (III.20) through (III.37), for the finite grids composed of $x \in [-u, u]$ and $K_{\mathrm{P},-} \in [0, K]$, finding a solution ($P_{\mathrm{A}}$, P_{B}, x, K_{P}) in terms of ($K_{\mathrm{P},-}$, x_-).

- Step 2: Compute the new approximations ζ^{n+1}, that is, the new coefficient vectors ξ^{n+1} by interpolating (P_{A}, P_{B}, x, K_{P}) on ($K_{\mathrm{P},-}$, x_-).

- Step 3: Check stopping criterion. If $\sup \|\zeta^{n+1} - \zeta^n\| \leq \delta$ then go to Step 4. Otherwise increase n by 1 and go to Step 1.

- Step 4: The algorithm terminates. Set $\zeta^* = \zeta^{n+1}$.

III.D Solutions of Four-Period Model Dynamics under Perfect Information

Table III.11 presents the equilibrium results under all possible paths of $\{u_t\}$.

Table III.11: Equilibrium solutions with all possible $\{u_t\}$ paths given perfect information.

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.7063	4.8825	8.7235	0.9073	TRUE	25.6415	4.1848	8.7235	1.0430	TRUE
2	0	23.8442	1.5715	13.6477	0.3224	FALSE	23.9191	1.5498	13.6966	0.3213	FALSE
3	0	16.9419	0	18.8244	0	FALSE	16.9951	0	18.8834	0	FALSE
4	0	0	0	25.4129	0	FALSE	0	0	25.4927	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.7063	4.8825	8.7235	0.9073	TRUE	25.6415	4.1848	8.7235	1.0430	TRUE
2	0	23.8442	1.5715	13.6477	0.3224	FALSE	23.9191	1.5498	13.6966	0.3213	FALSE
3	8	18.0756	4.5189	18.5759	0.3003	TRUE	18.1310	4.5328	18.6396	0.2990	TRUE
4	0	0	0	27.1134	0	FALSE	0	0	27.1966	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.7063	4.8825	8.7235	0.9073	TRUE	25.6415	4.1848	8.7235	1.0430	TRUE
2	0	23.8442	1.5715	13.6477	0.3224	FALSE	23.9191	1.5498	13.6966	0.3213	FALSE
3	12	19.6518	4.9130	18.2867	0.6501	TRUE	19.7131	4.9283	18.3549	0.6480	TRUE
4	0	0	0	29.4778	0	FALSE	0	0	29.5696	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.7063	4.8825	8.7235	0.9073	TRUE	25.6415	4.1848	8.7235	1.0430	TRUE
2	0	23.8442	1.5715	13.6477	0.3224	FALSE	23.9191	1.5498	13.6966	0.3213	FALSE
3	20	26.1563	6.5391	17.5092	1.5901	TRUE	26.2169	6.5542	17.5910	1.5845	TRUE
4	0	0	0	39.2345	0	FALSE	0	0	39.3253	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.7063	4.8825	8.7235	0.9073	TRUE	25.6415	4.1848	8.7235	1.0430	TRUE
2	8	25.9756	5.6813	13.0890	0.6325	TRUE	25.8276	4.8418	13.0996	0.7078	TRUE
3	0	18.4563	0	20.5071	0	FALSE	18.3512	0	20.3902	0	FALSE
4	0	0	0	27.6845	0	FALSE	0	0	27.5267	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.7063	4.8825	8.7235	0.9073	TRUE	25.6415	4.1848	8.7235	1.0430	TRUE
2	8	25.9756	5.6813	13.0890	0.6325	TRUE	25.8276	4.8418	13.0996	0.7078	TRUE
3	8	19.0488	4.7622	19.6517	0.2861	TRUE	19.0534	4.7634	19.6656	0.2843	TRUE
4	0	0	0	28.5732	0	FALSE	0	0	28.5801	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.7063	4.8825	8.7235	0.9073	TRUE	25.6415	4.1848	8.7235	1.0430	TRUE
2	8	25.9756	5.6813	13.0890	0.6325	TRUE	25.8276	4.8418	13.0996	0.7078	TRUE
3	12	20.0354	5.0088	18.4699	0.6813	TRUE	20.1882	5.0471	18.6901	0.6672	TRUE
4	0	0	0	30.0530	0	FALSE	0	0	30.2823	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.7063	4.8825	8.7235	0.9073	TRUE	25.6415	4.1848	8.7235	1.0430	TRUE
2	8	25.9756	5.6813	13.0890	0.6325	TRUE	25.8276	4.8418	13.0996	0.7078	TRUE
3	20	25.6611	6.4153	14.7223	1.9346	TRUE	26.0506	6.5127	15.7399	1.8249	TRUE
4	0	0	0	38.4917	0	FALSE	0	0	39.0760	0	FALSE

III.E Solutions of Four-Period Model Dynamics under Imperfect Information

Table III.12 presents the equilibrium results under all possible paths of $\{u_t\}$ with imperfection information.

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.7063	4.8825	8.7235	0.9073	TRUE	25.6415	4.1848	8.7235	1.0430	TRUE
2	12	27.0946	5.9725	12.5066	0.9558	TRUE	26.8581	5.0745	12.5502	1.06361	TRUE
3	0	19.2515	0	21.3905	0	FALSE	19.0834	0	21.2038	0	FALSE
4	0	0	0	28.8772	0	FALSE	0	0	28.6251	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.7063	4.8825	8.7235	0.9073	TRUE	25.6415	4.1848	8.7235	1.0430	TRUE
2	12	27.0946	5.9725	12.5066	0.9558	TRUE	26.8581	5.0745	12.5502	1.0636	TRUE
3	8	19.8220	4.9555	20.5472	0.2683	TRUE	19.7637	4.9409	20.4877	0.2681	TRUE
4	0	0	0	29.7329	0	FALSE	0	0	29.6455	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.7063	4.8825	8.7235	0.9073	TRUE	25.6415	4.1848	8.7235	1.0430	TRUE
2	12	27.0946	5.9725	12.5066	0.9558	TRUE	26.8581	5.0745	12.5502	1.0636	TRUE
3	12	20.8153	5.2038	19.3205	0.6585	TRUE	20.9085	5.2271	19.4774	0.6464	TRUE
4	0	0	0	31.2229	0	FALSE	0	0	31.3627	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.7063	4.8825	8.7235	0.9073	TRUE	25.6415	4.1848	8.7235	1.0430	TRUE
2	12	27.0946	5.9725	12.5066	0.9558	TRUE	26.8581	5.0745	12.5502	1.0636	TRUE
3	20	26.3473	6.5868	15.4548	1.8883	TRUE	26.6950	6.6737	16.4430	1.7825	TRUE
4	0	0	0	39.5209	0	FALSE	0	0	40.0424	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.7063	4.8825	8.7235	0.9073	TRUE	25.6415	4.1848	8.7235	1.0430	TRUE
2	20	32.0389	7.3092	10.9442	1.8229	TRUE	32.0392	6.2874	10.9027	2.1305	TRUE
3	0	22.7645	0	25.2939	0	FALSE	22.7647	0	25.2941	0	FALSE
4	0	0	0	34.1467	0	FALSE	0	0	34.1470	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.7063	4.8825	8.7235	0.9073	TRUE	25.6415	4.1848	8.7235	1.0430	TRUE
2	20	32.0389	7.3092	10.9442	1.8229	TRUE	32.0392	6.2874	10.9027	2.1305	TRUE
3	8	23.2132	5.8033	24.5565	0.1917	TRUE	23.3022	5.8256	24.6694	0.1888	TRUE
4	0	0	0	34.8198	0	FALSE	0	0	34.9534	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.7063	4.8825	8.7235	0.9073	TRUE	25.6415	4.1848	8.7235	1.0430	TRUE
2	20	32.0389	7.3092	10.9442	1.8229	TRUE	32.0392	6.2874	10.9027	2.1305	TRUE
3	12	24.2253	6.0563	23.1275	0.5631	TRUE	24.4869	6.1217	23.4811	0.5479	TRUE
4	0	0	0	36.3379	0	FALSE	0	0	36.7303	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.7063	4.8825	8.7235	0.9073	TRUE	25.6415	4.1848	8.7235	1.0430	TRUE
2	20	32.0389	7.3092	10.9442	1.8229	TRUE	32.0392	6.2874	10.9027	2.1305	TRUE
3	20	29.3671	7.3418	18.7466	1.7019	TRUE	29.9215	7.4804	20.0336	1.5897	TRUE
4	0	0	0	44.0507	0	FALSE	0	0	44.8823	0	FALSE

Table III.12: Equilibrium solutions with all possible $\{u_t\}$ paths given imperfect information.

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.6586	4.8850	8.7235	0.8971	TRUE	25.5934	4.1843	8.7235	1.0317	TRUE
2	0	23.7926	1.5319	13.6289	0.3180	FALSE	23.8660	1.5106	13.6767	0.3169	FALSE
3	0	16.9053	0	18.7836	0	FALSE	16.9574	0	18.8415	0	FALSE
4	0	0	0	25.3579	0	FALSE	0	0	25.4361	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.6586	4.8850	8.7235	0.8971	TRUE	25.5934	4.1843	8.7235	1.0317	TRUE
2	0	23.7926	1.5319	13.6289	0.3180	FALSE	23.8660	1.5106	13.6767	0.3169	FALSE
3	8	18.0446	4.5111	18.5410	0.3010	TRUE	18.0989	4.5247	18.6033	0.29968	TRUE
4	0	0	0	27.0669	0	FALSE	0	0	27.1484	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.6586	4.8850	8.7235	0.8971	TRUE	25.5934	4.1843	8.7235	1.0317	TRUE
2	0	23.7926	1.5319	13.6289	0.3180	FALSE	23.8660	1.5106	13.6767	0.3169	FALSE
3	12	19.6247	4.9062	18.2592	0.6505	TRUE	19.6847	4.9212	18.3260	0.6485	TRUE
4	0	0	0	29.4370	0	FALSE	0	0	29.5271	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.6586	4.8850	8.7235	0.8971	TRUE	25.5934	4.1843	8.7235	1.0317	TRUE
2	0	23.7926	1.5319	13.6289	0.3180	FALSE	23.8660	1.5106	13.6767	0.3169	FALSE
3	20	26.1361	6.5340	17.5023	1.5892	TRUE	26.1954	6.5489	17.5825	1.5837	TRUE
4	0	0	0	39.2042	0	FALSE	0	0	39.2932	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.6586	4.8850	8.7235	0.8971	TRUE	25.5934	4.1843	8.7235	1.0317	TRUE
2	8	25.9014	5.6621	13.0733	0.6263	TRUE	25.7572	4.8260	13.0830	0.7015	TRUE
3	0	18.4036	0	20.4484	0	FALSE	18.3012	0	20.3346	0	FALSE
4	0	0	0	27.6054	0	FALSE	0	0	27.4517	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.6586	4.8850	8.7235	0.8971	TRUE	25.5934	4.1843	8.7235	1.0317	TRUE
2	8	25.9014	5.6621	13.0733	0.6263	TRUE	25.7572	4.8260	13.0830	0.7015	TRUE
3	8	18.9974	4.7494	19.5924	0.2872	TRUE	19.0048	4.7512	19.6096	0.2854	TRUE
4	0	0	0	28.4961	0	FALSE	0	0	28.5072	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.6586	4.8850	8.7235	0.8971	TRUE	25.5934	4.1843	8.7235	1.0317	TRUE
2	8	25.9014	5.6621	13.0733	0.6263	TRUE	25.7572	4.8260	13.0830	0.7015	TRUE
3	12	19.9835	4.9959	18.4136	0.6828	TRUE	20.1389	5.0347	18.6364	0.6686	TRUE
4	0	0	0	29.9753	0	FALSE	0	0	30.2084	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.6586	4.8850	8.7235	0.8971	TRUE	25.5934	4.1843	8.7235	1.0317	TRUE
2	8	25.9014	5.6621	13.0733	0.6263	TRUE	25.7572	4.8260	13.0830	0.7015	TRUE
3	20	25.6156	6.4039	14.6738	1.9377	TRUE	26.0066	6.5017	15.6920	1.8278	TRUE
4	0	0	0	38.4233	0	FALSE	0	0	39.0099	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.6586	4.8850	8.7235	0.8971	TRUE	25.5934	4.1843	8.7235	1.0317	TRUE
2	12	27.0133	5.9512	12.4883	0.9508	TRUE	26.7815	5.0571	12.5321	1.0582	TRUE
3	0	19.1937	0	21.3263	0	FALSE	19.0290	0	21.1433	0	FALSE
4	0	0	0	28.7905	0	FALSE	0	0	28.5435	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.6586	4.8850	8.7235	0.8971	TRUE	25.5934	4.1843	8.7235	1.0317	TRUE
2	12	27.0133	5.9512	12.4883	0.9508	TRUE	26.7815	5.0571	12.5321	1.0582	TRUE
3	8	19.7659	4.94146372	20.4819408	0.26957766	TRUE	19.7110	4.92774665	20.4265031	0.2693186	TRUE
4	0	0	0	29.6487823	0	FALSE	0	0	29.5664799	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.6586	4.8850	8.7235	0.8971	TRUE	25.5934	4.1843	8.7235	1.0317	TRUE
2	12	27.0133	5.9512	12.4883	0.9508	TRUE	26.7815	5.0571	12.5321	1.0582	TRUE
3	12	20.7587	5.1897	19.2586	0.6602	TRUE	20.8551	5.2138	19.4187767	0.6479	TRUE
4	0	0	0	31.1381	0	FALSE	0	0	31.2826	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.6586	4.8850	8.7235	0.8971	TRUE	25.5934	4.1843	8.7235	1.0317	TRUE
2	12	27.0133	5.9512	12.4883	0.9508	TRUE	26.7815	5.0571	12.5321	1.0582	TRUE
3	20	26.2975	6.5744	15.4014	1.8916	TRUE	26.6471	6.6618	16.3906	1.7856	TRUE
4	0	0	0	39.4462	0	FALSE	0	0	39.9707	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.6586	4.8850	8.7235	0.8971	TRUE	25.5934	4.1843	8.7235	1.0317	TRUE
2	20	31.9534	7.2854	10.9137	1.8243	TRUE	31.9577	6.2678	10.87427	2.1320	TRUE
3	0	22.7038	0	25.2264	0	FALSE	22.7068	0	25.2298	0	FALSE
4	0	0	0	34.0556	0	FALSE	0	0	34.0602	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.6586	4.8850	8.7235	0.8971	TRUE	25.5934	4.1843	8.7235	1.0317	TRUE
2	20	31.9534	7.2854	10.9137	1.8243	TRUE	31.9577	6.2678	10.87427	2.1320	TRUE
3	8	23.1548924	5.7887	24.4864	0.1930	TRUE	23.2470	5.8117	24.6030	0.1900	TRUE
4	0	0	0	34.7323	0	FALSE	0	0	34.8705	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.6586	4.8850	8.7235	0.8971	TRUE	25.5934	4.1843	8.7235	1.0317	TRUE
2	20	31.9534	7.2854	10.9137	1.8243	TRUE	31.9577	6.2678	10.87427	2.1320	TRUE
3	12	24.1668	6.0417	23.0610	0.5647	TRUE	24.4311	6.1078	23.4175	0.5493	TRUE
4	0	0	0	36.2501	0	FALSE	0	0	36.6467	0	FALSE

Period	u_t	K Loose	X Loose	C^{IM} Loose	Spread	Bind.	K Tight	X Tight	C^{IM} Tight	Spread Tight	Bind. Tight
1	8	25.6586	4.8850	8.7235	0.8971	TRUE	25.5934	4.1843	8.7235	1.0317	TRUE
2	20	31.9534	7.2854	10.9137	1.8243	TRUE	31.9577	6.2678	10.87427	2.120	TRUE
3	20	29.3151	7.3288	18.6890	1.7049	TRUE	29.8709	7.4677	19.9765	1.5925	TRUE
4	0	0	0	43.9727	0	FALSE	0	0	44.8064	0	FALSE

Chapter IV

Arbitrage, Financial Accelerator, and Sudden Market Freezes

We develop an infinite horizon model that links the intermediation in both the financial and real sectors. Intermediaries provide market liquidity and exploit the arbitrage profits in segmented financial markets. To do so, they use their productive capital as collateral. We show that the weakened intermediation and arbitrage losses are mutually reinforcing during an economic downturn. This forces intermediaries to de-lever and leads to liquidity spirals in both financial and real sectors. Also, the distress might further open up the possibility of sudden run-like market freezes, where intermediaries are denied access to renewed funding through arbitrage. We evaluate the effect of three intervention policies: direct purchase of distressed assets, interest rate cuts, and capital injection. We find that capital injection is most effective as it loosens the margin requirement. The interest cut is least effective because it exacerbates the capital misallocation.

Keywords: limit of arbitrage, financial intermediary, haircut, segmented markets, financial crises, market liquidity, collateral constraints

JEL Classification: D52 D58 E44 G01 G12

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IV.1 Introduction

There are two distinct approaches in the theoretical literature to capture the relationship between limits of arbitrage and the financial intermediaries, as well as the crises caused by intermediation failure. The first, in the finance literature and pioneered by Shleifer and Vishny (1997), focuses on how financial frictions hinder arbitrageurs' capability to fully eliminate the mispricings in financial markets. A contraction in the arbitrageurs' capital weakens their ability to speculate in the price anomalies. In turn, this might further exacerbate mispricings, trigger the liquidity and loss spirals, and eventually lead to the market collapse. The other approach, summarized by Gertler et al. (2016) in the macroeconomic literature, emphasizes how financial frictions give rise to limits of arbitrage reflected in the real sector via the credit cycle. Fueled by high leverage ratios, tiny negative shocks can propagate into heavy contractions in borrower's wealth, further depressing collateral price and raising the credit spreads. Moreover, liquidity mismatch can also render banks vulnerable to the threat of runs.

Nonetheless, less attention has been paid to the interactions between limits of arbitrage in the financial markets and that in the real sector. We aim to fill the gap by connecting the arbitrage trading in the financial market and the intermediation in the real sector via collateral constraints. We are also interested in providing an alternative perspective on explaining both the linear and non-linear aspects of the market collapse.

Rather than modeling the market collapse through explicit credit cycles, our approach focuses on the intermediation failure associated with arbitrage trading. The market freezes when all arbitrageurs become insolvent and are denied access to renewed funding. Put it differently, the whole intermediary sector is excluded from exploiting arbitrage profits and from intermediation of productive capital altogether. As a result, the market liquidity dries up in both financial and real sectors. Our motivation draws heavily from the recent episodes. During the 1998 hedge fund crisis and the 2007–2009 subprime crisis, many quantitative broker-dealers, who had engaged in various arbitrage trade strategies, suffered huge losses. They were then haunted by margin calls and forced into fire sales. For example, in March of 2008, Bear Stearns' counterparties altogether terminated their brokerage relationships and ran on it. In September 2008, several counterparties requested additional collateral from AIG for its credit default swap positions. Had the Fed not intervened, such requests would have driven the firm out of market. In October 2008, almost all the major quant funds and investment banks suddenly failed, credit spreads soared, and markets for many financial securities froze. The distress quickly spilled over to the real sector. Though absent from any significant adverse shocks, the output and the real investment experienced an eight and a forty percent sharp contraction, respectively.

To capture this phenomenon, we develop a simple model to incorporate the arbitrage trading and the real activities, which features financial accelerator effects, market co-movement and sudden self-fulfilling market freezes. As a complementary approach to modeling the transmission channel of the financial distress, we focus on the pecuniary externality when arbitrageurs unwind their trading positions. We show that such externality can generate a much more amplified contraction in arbitrageurs' budget during an economic downturn, triggering the liquidity to dry up across markets and opening the door for self-fulfilling crises. As a result, a very moderate leverage ratio can produce strong financial accelerator effects and render the economy susceptible to run-like market freezes.

In particular, we consider an infinite horizon, small and open economy with a fixed supply of capital and two segmented financial markets. There are two types of price-taking agents: the household investors and the intermediaries. Similar to Gromb and Vayanos (2017), household investors live in segmented markets and they are subject to negatively correlated endowment shocks. Thus, households in different markets have opposite hedging demands to smooth their consumptions, giving rise to price discrepancies between identical financial securities. The intermediaries, on the other hand, assume dual roles in our model. On one hand, they are arbitrageurs who can trade in both segmented financial markets. While exploiting the price spreads, they also provide liquidity to each market. Meanwhile, they are more efficient in managing productive capital than the households. Moreover, agents can use capital as collateral in the financial markets.

Due to the margin requirement, the market liquidity in the financial markets, measured by intermediaries' arbitrage capability, affects and is affected by the market liquidity of capital in the real sector. An initial contraction in net worth during an economic downturn forces intermediaries to sell some of their collateral, depressing the capital price and increasing the haircut rate. This in turn tightens the margin requirement and further hinders intermediaries' capability to exploit arbitrage profits. This is because with less collateral, they have to unwind some of their arbitrage positions. As intermediaries fail to internalize the price effects when they collectively do so, they also suffer losses on their initial positions. Thus, the worsening balance sheet condition triggers more fire sale of capital and aggressive unwinding. While intermediaries' financial losses reinforce their weakened intermediation capacity, the market liquidity quickly dries up in both financial and capital markets.

Similar to Gertler and Kiyotaki (2015), apart from the recession caused by the procyclical movement of market liquidity, crises can also occur as a sunspot phenomenon. Due to liquidity mismatch in different sectors, a sudden market freeze may be possible. The existence of such an equilibrium relies on three critical factors: 1) intermediaries' net worth; 2) the liquidation capital price and 3) asset price spreads. A heavy portfolio loss can render intermediaries vulnerable to sudden bankruptcy during a market freeze, though they are immune to funding failure in normal times.

Our specific modeling of the sudden market freeze follows the approach as in Cole and Kehoe (2000)'s self-fulfilling debt crises and Gertler and Kiyotaki (2015)'s bank runs. In these papers, the sunspot self-fulfilling crises arise from a panic failure to roll over short-term debt. The market freeze here, however, reflects a denial of intermediaries' access to arbitrage trading as a critical part of their funding sources.

We also use the model as a laboratory to quantitatively evaluate government interventions. Starting from the state shortly after the market freeze, we track the recovery paths conditional on three common intervention policies: (i) capital injection by infusing equity or lending loans to the intermediaries; (ii) direct purchase of capital by the government; (iii) lowering the interest rate. In our experiment, we compare the direct capital purchase to the same amount of equity infusion. As in He and Krishnamurthy (2013), we find that equity infusion is much more effective in stimulating the economy and helping the intermediary sector to recover faster. This is because part of the friction in our model comes from the constraint on intermediaries' net worth. Accordingly, direct funding support to intermediaries can most effectively restore the overall efficiency and revive the economy.

By acting as buyers in the capital market, the government helps mitigate the sharp contraction in aggregate output, raise the capital price and in turn alleviate the margin requirement in the financial markets. However, this comes at the cost of slowing down the recapitalization of the intermediary sector in the long run, as the government purchase partially crowds out intermediaries in the capital market by raising their marginal cost.

Comparably, we find that the interest rate policy has rather limited stimulating effects. Except for moderately raising the capital price at the beginning, lowering the interest rate overall exacerbates the inefficiency in asset allocation and slows down the recovery process as a whole. This occurs because in our model the leverage of intermediaries does not rely on the credit channel explicitly. The interest cut instead leads the less efficient households to flood the capital market, pushing out intermediaries and thus hindering their capability to recapitalize.

The key contribution of our paper is to work out an equilibrium model linking intermediation in both financial and real sectors, and is dynamic, parsimonious and tractable. It complements a growing theoretical literature on the limits of arbitrage, and especially to the strand focusing on the arbitrageurs' margin requirement. We contribute to this literature by associating the collateral constraints with intermediation in the real sector.

Shleifer and Vishny (1997) are the first to study how trading restrictions may affect arbitrageurs' capability to correct mispricing. Due to frictions arising from the asymmetric information and moral hazard, arbitrageurs bear insolvency risk under the margin requirement.

Our setup of arbitrage trading borrows heavily from Gromb and Vayanos (2002, 2017). They develop a general equilibrium model in which collateral constrained arbitrageurs intermediate trade across segmented markets. Our main departure from these models is that we allow for a broader range of assets (as opposed to only the riskless asset) to serve as collateral in the financial market. This enables us to study the spillover effects between the financial and real sector. Also, instead of emphasizing the self-correcting dynamics and cross-sectional relationship as in Gromb and Vayanos (2017), our model stresses the amplification and spillover effects, as well as the nonlinear nature of intermediation failure.

Our paper also shares similarities with many other models featuring financially constrained arbitrageurs. To name a few, Brunnermeier and Pedersen (2009) study the feedback loops of arbitrageurs' funding liquidity and market liquidity, and how they interact through the collateral constraints. There the funding liquidity captures the arbitrageurs' capability of raising debt to facilitate the arbitrage trading. Our principle difference lies in the source and the objective of arbitrageurs' funding. In our model, the major funding comes from arbitrage profits rather than from direct borrowing, and it is used to acquire productive capital. Hence, the funding liquidity is reflected by the market liquidity of financial assets. In He and Krishnamurthy (2012, 2013), arbitrageurs can raise funds from other investors to invest in a risky financial security, but this external funding cannot exceed a given ratio of their own wealth. Consistent with our policy implication, He and Krishnamurthy (2013) demonstrate that ex post equity injection is a superior policy compared to interest rate cuts or asset purchasing programs by the government. Liu and Longstaff (2004) study the optimal arbitrage strategy of risk-averse, collateral-constrained arbitrageurs in a partial equilibrium. Xiong (2001) and Kyle and Xiong (2001) examine the impact of arbitrage capital on asset prices by analyzing the wealth effects of arbitrageurs with log utility in a continuous-time model. However, the above papers do not study the inefficiency caused by capital misallocation.

In addition, our paper is also related to macro models stressing the financial accelerator effects. (See e.g., Bernanke and Gertler (1989); Kiyotaki and Moore (1997); Bernanke et al. (1999); Kocherlakota (2000); Gertler et al. (2012); Gertler and Kiyotaki (2015); Gertler et al. (2016)). These models highlight the feedback loop from the fall in collateral prices to a fall in credit capacity. Key to this mechanism is the role of leverage: the exposure of balance sheets to systemic risk is increasing in the degree of borrowers' leverage. We differ by identifying the price externality of the arbitrage activities. Instead of facing fixed debt commitment, intermediaries suffer increasing mark-to-market losses when they collectively unwind their arbitrage positions. In this respect, some recent works also underscore similar externality, as borrowers do not internalize the impact of their own leverage decisions on the systemic risk. Examples include Lorenzoni (2008); Bianchi (2011); Chari and Kehoe (2016); and Brunnermeier and Sannikov (2014).

Our paper also joins the literature characterizing the nonlinear nature of financial crises. One strand features the occasionally binding borrowing constraints as the source of nonlinearity. Examples are Brunnermeier and Sannikov (2014); He and Krishnamurthy (2014); Mendoza (2010). In these models, a negative shock might shift the economy into a crisis state with binding credit constraints, in which shocks get amplified and the intermediation is disrupted. Another stream to address the nonlinearity is through network interactions, e.g., Gârleanu et al. (2015). Yet another approach is to model the discrete and sudden nature of crises as an abrupt cease of financing possibilities, such as bank runs. Pioneered by Diamond and Dybvig (1983), recent work includes Cole and Kehoe (2000); Ennis and Keister (2003); Uhlig (2010); Gorton and Metrick (2012); Angeloni and Faia (2013); Martin et al. (2014); Gertler and Kiyotaki (2015); Boissay et al. (2016) and Gertler et al. (2016, 2017). Our modeling of market freezes belongs to this approach. Both the externality and the sudden stop of arbitrage trading provide the source of the nonlinearity. Different from Gertler and Kiyotaki (2015) and Gertler et al. (2016, 2017), the externality in our model gives rise to a procyclical optimal maximum leverage ratio,

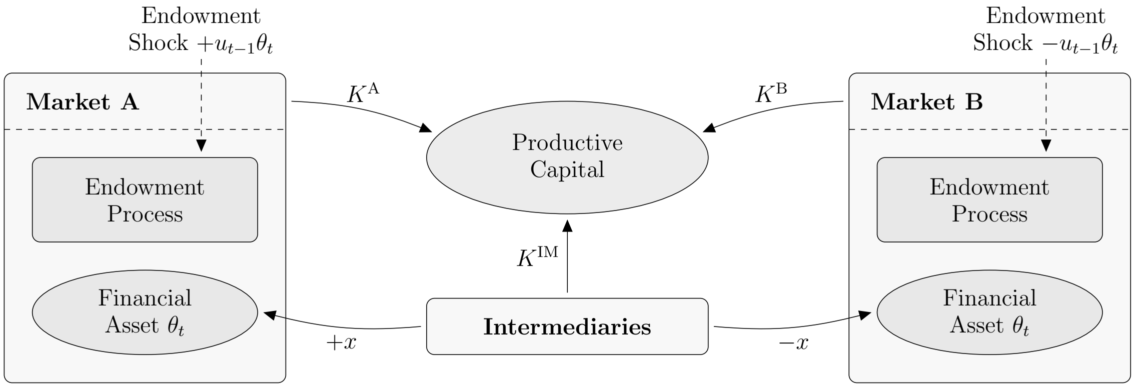

Figure IV.1: The structure of the economic system.

which tightens the margin constraints in a downturn and thus strengthens the nonlinear effects.

IV.2 Baseline Model

Figure IV.1 lays out the building blocks of our model. We consider a discrete time, infinite horizon, small and open economy with two segmented markets, market A and B. There are two types of competitive agents: household investors within each single market and specialized intermediaries. Each constitute a continuum of unit measure.

Within each segmented market, there exists an identical, long-lived risky financial asset, which pays out dividends in every period and is each in zero net supply. We identify them in different markets as asset A and B. Households in market A (B) can only trade asset A (B). As will become clear later, households' demands for the risky asset is opposite in market A and B. Besides, all agents have access to a riskless financial asset with an exogenous return $r > 0$.

In the production sector, there are two types of goods: the perishable one as numéraire, and the durable one as productive capital. The capital doesn't depreciate and is fixed in total supply, which we normalize to be unity. Both intermediaries and households can hold and operate on the capital.

Intermediaries can exploit arbitrage profits through intermediating between segmented markets. They reinvest part of the profits to acquire the capital. As opposed to studying the mechanism of intermediation in the credit cycle like those seminal papers such as

Bernanke and Gertler (1989) and Kiyotaki and Moore (1997), we are centrally interested in the intermediary channel of speculative security trading.

IV.2.1 Risky Assets

The risky assets that we model here is similar to the ones in Gromb and Vayanos (2017). In period t, it pays out a dividend d_t:

$$d_t = \bar{d} + \theta_t. \tag{IV.1}$$

$\bar{d}$ is a positive constant, and θ_t is a sequence of independent identically distributed (i.i.d) random variables, whose probability density function is symmetric around zero over the bounded support $\left[-\bar{\theta}, \bar{\theta}\right]$. We denote P_t^i as the ex-dividend price of asset i in period t, $i \in \{\mathrm{A}, \mathrm{B}\}$. The asset's risk premium in period t is given by

$$\psi_t^i = \frac{\bar{d}}{r} - P_t^i, \tag{IV.2}$$

the present value of expected future dividends discounted at the riskless rate r, minus the current price.

As in Gromb and Vayanos (2017), the assumption of identical payoffs is for expositional convenience. The aim is to reflect the real world mispricings among assets or portfolios of similar payoffs, yet traded at significantly different prices. Examples include (i) bonds and their corresponding CDS, (ii) Siamese-twin stocks, traded in different exchanges with identical dividends, (iii) Chinese A and B shares of stocks, (iv) the two legs of covered interest arbitrage strategies in the currency markets. The modeling of bounded support of θ_t is to facilitate the derivation of the collateral constraint on which we will elaborate later. Further, the zero net supply assumption is to ensure that intermediaries will hold opposite positions in the risky asset across markets and are thus immune from the randomness of the asset payoffs.

IV.2.2 Production

We adopt a similar setting of productivity differences between intermediaries and households as in Gertler and Kiyotaki (2015). In particular, we assume that when one intermediary operates on k_t^{IM} units of capital in period t, there is a payoff of $Z_{t+1}k_t^{\mathrm{IM}}$ units of consumption goods in period $t+1$ plus the residual capital, where Z_{t+1} is a multiplicative aggregate shock to productivity.

$$
\begin{array}{ll}
\text{date } t & \qquad\text{date } t+1 \\
\left.k_t^{\mathrm{IM}} \quad \text{capital}\right\} & \rightarrow \begin{cases} Z_{t+1}k_t^{\mathrm{IM}} & \text{output,} \\ k_t^{\mathrm{IM}} & \text{capital.} \end{cases}
\end{array}
$$

We also suppose that when the households invest k_t^i units of capital at t for a payoff at $t+1$, they are facing a management cost of $f\left(Q_t k_t^i\right)$ units of perishable goods at t, where Q_t is the capital price. Specifically,

$$
\begin{array}{ll}
\text{date } t & \qquad\text{date } t+1 \\
\left.\begin{array}{ll} k_t^i & \text{capital} \\ f\left(k_t^i\right) & \text{goods} \end{array}\right\} & \rightarrow \begin{cases} Z_{t+1}k_t^i & \text{output,} \\ k_t^i & \text{capital.} \end{cases}
\end{array}
$$

The management cost is intended to capture the households' lack of competence relative to intermediaries, such as commercial and investment banks or hedge funds, in screening and monitoring investment projects. Or they are not as specialized or sophisticated investors as those financial experts in analyzing and deciphering the complicated securitized products. Further, we assume that for household investors, the management cost is increasing and convex in the total value of their capital holdings:

$$
f(Q_t, k_t^i) = \frac{a}{2}Q_t(k_t^i)^2,
$$

with $a > 0$. The cost function implies that households' management burden increases with the price and the size of their capital. Thus, it becomes more expensive at the margin for households to manage capital directly.

As explained in Gertler and Kiyotaki (2015), without financial frictions, the more efficient intermediaries will manage all the capital stock and households do not directly invest in production. However, if the intermediaries cannot afford to finance the purchase of all capital, households will directly hold some of it. Moreover, to the extent that the intermediaries' purchasing capability becomes more constrained in a recession, as will be the case in our model, households' total share of capital will expand.

IV.2.3 Households

Households are competitive and infinitely lived. They consume in each period and have negative exponential utility. They save by investing in financial assets and capital. In period t, a typical household in market i chooses positions $\left\{y_s^i\right\}_{s\geq t}$ and $\left\{B_s^i\right\}_{s\geq t}$ in the risky and riskless securities, the capital holdings $\left\{k_s^i\right\}_{s\geq t}$ and consumption $\left\{c_s^i\right\}_{s\geq t}$ to maximize

$$-\mathbb{E}_t\left[\sum_{s=t}^{\infty}\beta^{s-t}\exp\left\{-\alpha c_s^i\right\}\right],$$

where α is the coefficient of absolute risk aversion and β is the subjective discount factor. The assumption of negative exponential utility rules out wealth effects on asset pricing and facilitates our later analysis.

The market segmentation of the risky assets simply means that households in market A cannot trade asset B directly with those in market B and vice versa. This assumption is to reflect the high cost faced by investors in different countries or different markets, due to regulatory constraints or informational asymmetries.

Similar to the settings in Gromb and Vayanos (2017), we assume that households in markets A and B receive random endowments, which affect their demand to hold risky assets. In period t, households in market i receive an endowment shock equal to

$$u_{t-1}^i\theta_t, \quad i\in\{\mathrm{A},\mathrm{B}\},$$

where u_{t-1}^i is known in period $t-1$. We refer u_t^i as the shock intensity and the random variable θ_t as the shock unit. As u_t^i is always revealed one period earlier than the shock unit, households know their hedging demands in advance.

To ensure that the demands of asset A and B differ and thus also their prices, we assume $u_t := u_t^{\mathrm{A}} = -u_t^{\mathrm{B}} > 0$ for expositional convenience. This assumption, together with that of zero net supply, simplifies our analysis by ensuring that intermediaries' positions in asset A and B are opposite.

As $u_t^{\mathrm{A}} > 0$, households in market A receive an endowment which is positively correlated with the shock unit θ_t and hence with asset A's payoff. Therefore, asset A becomes riskier and less desirable for households in market A. By contrast, asset B becomes more attractive for local households. In the absence of market segmentation, households in the two markets can directly trade u_t units of risky assets with each other, realizing full risk sharing and price parity. However, the market segmentation implies that without further intermediation the price of asset A is lower than that in the full risk sharing case, while asset B's price is higher. This leads to a potential arbitrage opportunity for intermediaries who can trade risky assets in both markets. They can lock in profits by longing asset A and shorting asset B. In doing so, intermediaries essentially provide liquidity to households as they allow them to hedge.

Our assumptions regarding the risky asset and household investors aim to resemble real-world examples. For instance, in the bond markets, the household investors can represent those institutions, which are obliged to hold bonds with particular coupon rates and times to maturity, such as pension funds or insurance companies. Their specific demands could stem from asset-liability management purpose or tax considerations. Similarly in the equity markets, domestic-equity mutual funds are often constrained to invest only in local stock markets. Domestic investment inflows and outflows might determine their asset demands, which correspond to our endowment shocks, thus giving rise to significant demand and price discrepancies in similar stocks traded in different countries.

Accordingly, households in market i, $i \in \mathrm{A}, \mathrm{B}$, are subject to the following budget constraint.

$$\begin{aligned} c_t^i = & \underbrace{y_{t-1}^i \left(d_t + P_t^i\right)}_{\substack{\text{value and dividend} \\ \text{of risky asset} \\ \text{holdings}}} + \underbrace{\left(Z_t + Q_t\right) k_{t-1}^i}_{\substack{\text{value and output} \\ \text{of capital holdings}}} + \underbrace{(1+r) B_{t-1}^i}_{\substack{\text{return from} \\ \text{the riskless} \\ \text{asset}}} + \underbrace{u_{t-1}^i \theta_t}_{\substack{\text{endowment} \\ \text{shocks}}} \\ & - \underbrace{y_t^i P_t^i - Q_t k_t^i - B_t^i}_{\substack{\text{investment on} \\ \text{financial assets} \\ \text{and capital}}} - \underbrace{\frac{a}{2} Q_t \left(k_t^i\right)^2}_{\substack{\text{management} \\ \text{cost}}} . \end{aligned} \tag{IV.3}$$

Here, consumption, saving, investment and the management cost are financed by the endowment as well as returns on the saving and investment from previous period.

Thus, the first order condition for riskless assets is given by

$$\mathbb{E}_t \left[\Gamma_{t,t+1}\right] (1+r) = 1, \tag{IV.4}$$

where the stochastic discount factor $\Gamma_{t,t+1}$ satisfies

$$\Gamma_{t,t+1} = \beta \exp\left\{-\alpha\left(c_{t+1}^i - c_t^i\right)\right\}. \tag{IV.5}$$

Given all agents have perfect foresight of aggregate productivity $\{Z_s\}_{s\geq t}$ in period t in our baseline model, the first order condition for direct capital holdings is given by

$$\mathbb{E}_t\left[\Gamma_{t,t+1}R_{K,t+1}\right] = 1 \tag{IV.6}$$

with

$$R_{K,t+1} = \frac{Z_{t+1} + Q_{t+1}}{Q_t + f_K'\left(Q_t, k_t^i\right)} = \frac{Z_{t+1} + Q_{t+1}}{Q_t\left(1 + ak_t^i\right)} = 1 + r,$$

where $R_{K,t+1}$ is the households' gross marginal rate of return from direct capital holdings. In equilibrium it should be equal to the return of the riskless assets.

Note that as long as the households have some positive share of the capital, the first order condition (IV.6) will help determine the market price of capital. Also, the capital price tends to decrease in the households' capital share as the marginal cost $f_K'(Q_t, k_t^i)$ is increasing. As we will discuss later, in the wake of financial distress, intermediaries are forced to liquidate their capital, which will further depress the price. In the limiting case of a total collapse of the intermediary sector, households absorb all the capital and the price slumps to a minimum level.

Likewise, the first order condition for the risky asset i is given by

$$\mathbb{E}_t\left[\Gamma_{t,t+1}\frac{P_{t+1}^i + d_{t+1}}{P_t^i}\right] = 1. \tag{IV.7}$$

We denote the excess return per share of asset i ($i \in \{\mathrm{A}, \mathrm{B}\}$) relative to the riskless asset as

$$R_{t+1}^i \equiv d_{t+1} + P_{t+1}^i - (1+r)P_t^i = (1+r)\psi_t^i - \psi_{t+1}^i + \theta_{t+1},$$

where the second step follows from (IV.1) and (IV.2). In our baseline model, we assume that agents also have perfect foresight of the shock intensity $\{u_s^i\}_{s\geq t}$ and thus the risk premia are deterministic. For simplicity, we refer to

$$\Phi_t^i \equiv (1+r)\psi_t^i - \psi_{t+1}^i,$$

as the expected excess return of asset i and we present a more revealing first order condition for the risky assets below.

Proposition IV.1. *The household's optimal position in asset i is given by the first order condition*

$$\bar{\theta} g'\left(\left[u_t^i + y_t^i\right]\bar{\theta}\right) = \Phi_t^i, \tag{IV.8}$$

where the function g(x) is defined by

$$\exp\left\{\alpha g(x)\right\} \equiv \mathbb{E}\left[\exp\left\{-\frac{\alpha x \theta_t}{\bar{\theta}}\right\}\right].$$

The function $g(x)$ is non-negative, symmetric around the vertical axis, and strictly convex, with $\lim_{x\to\infty} g'(x) = 1$.

If we view the expected excess return of asset i, Φ_t^i as the marginal benefits of risk-taking, we can conclude that the position y_t^i is increasing in the asset's expected excess return Φ_t^i, since the function $g(x)$ is convex. In addition, as implied by (IV.8), when the risk premia or the expected excess returns in two markets are opposite, then the optimal positions for households in market A are also opposite to those in market B. This follows from the symmetric endowment shocks and that $g'(x) = -g'(-x)$ as implied by Proposition IV.1.

IV.2.4 Intermediaries

Unlike the households, the intermediaries can invest in the risky assets simultaneously in both markets. This assumption, together with them being more efficient in managing capital, capture the idea that intermediaries are more sophisticated than other investors. They correspond best to those financial institutions, which are involved in a wide spectrum of investment opportunities, ranging from trading complex financial assets to providing loans to productive manufacturers. For example, hedge funds, which are less subject to the informational or regulatory frictions contributing to market segmentation, or investment banks that both trade diverse financial assets and are involved in the shadow banking system. As intermediaries can invest in all assets, they are uniquely able to exploit price discrepancies between markets and provide liquidity to households.

We assume that intermediaries are competitive and risk neutral. Intermediaries fund capital investments by exploiting the arbitrage profits as well as by using their own net worth n_t. Due to financial market frictions, intermediaries may be constrained in their ability to eliminate arbitrage opportunities.

As pointed out by Gertler and Kiyotaki (2015), to the extent that intermediaries may face frictions to gain external funding, they will attempt to save their way out of funding hurdles by accumulating retained earnings in order to achieve 100 percent self-financing. To rule out this uninteresting case, we follow Bernanke et al. (1999) by assuming that intermediaries have a finite expected lifetime. In particular, whether an intermediary would exit at t follows a geometric distribution with a survival probability of $\sigma \in (0,1)$. The expected lifetime of an intermediary is thus $\frac{1}{1-\sigma}$.

To fill the void left by the exiting intermediaries, in every period new ones enter with a constant endowment w^{IM}, which is received only in the first period of their life. The number of entering intermediaries is equal to the number of exiting ones. This setup provides a simple way to generate a compulsory "dividend payout" from the intermediary sector in order to ensure that intermediaries are financially constrained in equilibrium.

Specifically, we assume that intermediaries draw utility from consumptions in the very period they exit. The expected utility of a surviving intermediary at the end of period t is thus given by

$$\mathbb{E}_t\left[\sum_{s=1}^{\infty}\beta^s(1-\sigma)\sigma^{s-1}c_{t+s}^{\mathrm{IM}}\right],$$

where $(1-\sigma)\sigma^{s-1}$ is the probability of exiting in period $t+s$, and c_{t+s}^{IM} is the terminal consumption if the intermediary exits at that time.

Figure IV.2 shows the timing. The aggregate shock Z_t is realized at the start of t. Conditional on this shock, the net worth of surviving intermediaries is given by the gross return on assets minus their liability from arbitrage trading:

$$n_t = (Z_t + Q_t)\, k_{t-1}^{\mathrm{IM}} + \sum_{i\in\{\mathrm{A},\mathrm{B}\}} x_{t-1}^i P_t^i,$$

where k_t^{IM} is the intermediary's capital holding at the end of period t.

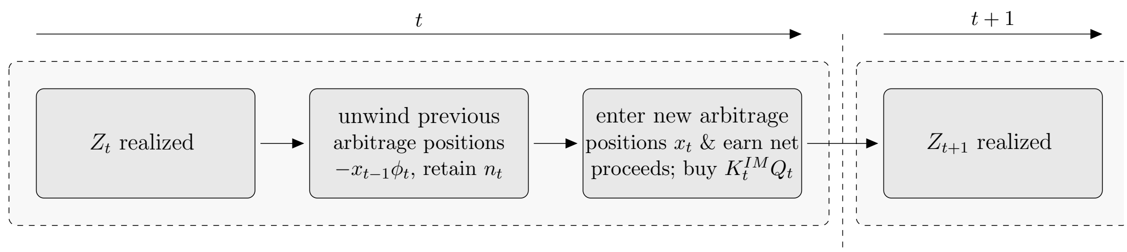

Figure IV.2: Timing

To simplify our analysis, we assume that intermediaries will encounter high management cost if they fail to take a net zero position in the risky assets. That is, they will have to take opposite positions with identical size in the two markets, i.e., $x_t^{\mathrm{A}} = -x_t^{\mathrm{B}}$. This assumption follows naturally when intermediaries are risk-averse. As will become clear later, even with risk neutral intermediaries, when their collateral constraints are binding, it is also not optimal for them to take unbalanced positions, as they will either "waste" collateral or induce inefficient spending on the purchase of risky assets instead of capital. Taking an extra unit of unbalanced risky asset will yield an expected return $1+r$, whereas purchasing capital will generate overall much higher returns.

Denote the price difference as $\phi_t \equiv P_t^{\mathrm{B}} - P_t^{\mathrm{A}}$, and the intermediary's risky position size as $x_t \equiv x_t^{\mathrm{A}}$. Then we can simplify the above as

$$n_t = (Z_t + Q_t)\, k_{t-1}^{\mathrm{IM}} - x_{t-1}\phi_t.$$

For the new born intermediaries at t, the net worth equals the initial endowment:

$$n_t = w^{\mathrm{IM}}.$$

On the other hand, exiting intermediaries sell their capital and cease trading risky assets. They then consume all their net worth:

$$c_t^{\mathrm{IM}} = n_t.$$

In each period t, a continuing intermediary, new or surviving, finances capital holdings $Q_t k_t^{\mathrm{IM}}$ with arbitrage income and the net worth:

$$Q_t k_t^{\mathrm{IM}} = - \sum_{i \in \{\mathrm{A,B}\}} x_t^i P_t^i + n_t = x_t \phi_t + n_t.$$

IV.2.5 Collateral Constraints

Collateral constraints arise in our model because agents need to collateralize their short positions in financial assets. This is consistent with observations from real-world markets. In general, restrictions on trading strategies are imposed through the standard requirement that investors provide collateral as security for their short positions. In particular, once a party incurs a liability by either shorting an asset or borrowing cash (which is essentially shorting the riskless asset), the central trading institutions usually require them to post collateral against mark-to-market losses.

Consider an agent who wants to enter a short position in a risky asset. The agent has to first borrow the asset so that he can sell it afterwards. In order to do so, he must post collateral to ensure that the asset loan will be repaid. We assume that agents are obliged to deposit enough collateral to guarantee they would honor any liabilities that their positions may generate, and do so in each market separately. In addition, we impose that intermediaries cannot use their long positions in the risky asset in one market as collateral for their short positions elsewhere. This is in the spirit of market segmentation: the same informational or regulatory frictions that prohibit households in market A from investing in asset B could also prohibit intermediaries' counterparties in market A from accepting asset B as collateral. Thus, we assume that agents can only use capital holdings or riskless assets as collateral. As we set $\mathbb{E}[Z_t] > r$, it is not optimal for intermediaries to hold riskless asset and hence they will end up only using capital as collateral. Households might use both. But for convenience, we assume that they will first resort to capital, which is preferred by intermediaries, before using the riskless assets. We also assume that households have enough wealth to collateralize any position they may want to establish, i.e., up to the magnitude of their shock intensity u_t^i. In this way, we are spared from discussing the situation where households are financially constrained to trade risky assets. However, the intermediaries might be constrained by their net worth.

To illustrate the collateral constraints, we consider an intermediary who intends to establish a short position in asset i. To capture the economics of collateralization in a simple, yet realistic way, we assume that the intermediary must post enough capital holdings such that her counterparty has collateral value at least equal to the amount owed by her plus a precautionary dividend deposit $\bar{\theta}$ per unit of position. Thus, if she enters one unit of short position in asset i, and gains the proceeds of P_t^i, then she must post capital holdings with collateral value in the amount of $P_t^i + \bar{\theta}$. While our modeling of including $\bar{\theta}$ in the collateral constraint is out of technical convenience, i.e., to avoid computing multiple equilibria, it can also well be interpreted as the asset i's specific margin requirement or as a precautionary measure against potential default on the uncertain dividend in the next period.

Also, we implicitly assume that if one party defaults, the counterparty can only seize the capital without the output in that period. This is consistent with the limited liability practice, in the sense that the output is essentially part of the debtor's labor income and is protected against creditors. Moreover, we assume that while the ownership of capital can be posted to the counterparty as collateral, the actual operation or management is still run by the party who enters the short position and thus we exclude the potential loss from moral hazard.

To account for the illiquidity of the capital relative to the riskless asset, we also model the margin requirement. In particular, the difference between the capital price and the collateral value per unit is denoted as the margin, or the haircut. We assume that the size of the required haircut increases with the market illiquidity of the capital. Specifically, an intermediary with x_t unit of short positions in asset i at t has to post "effective" collateral up to $P_t^i + \bar{\theta}$, after discounting by the haircut rate:

$$(1 - m_t)\, Q_t k_t \geq x_t \left(P_t^i + \bar{\theta}\right).$$

Here, $1 - m_t \equiv g + hQ_t \in (0, 1]$ is the margin or haircut rate, with $g > 0$ and $h > 0$. Note that we use the capital price Q_t as a measure of the market liquidity of capital. This is because Q_t increases with intermediaries' collective share of aggregate capital and reflects their demand. Thus, when Q_t is high, households find it easier to sell the capital to an intermediary who appreciates it more than household investors do, relative to the riskless asset. By establishing the mapping of the prevailing capital liquidity to the margin requirement, we intend to capture the real-world effect that the haircut tightens when the market of collateral becomes more illiquid.

Likewise, when an intermediary enters a long position in asset i, her counterparty also has to transfer collateral to her. As will become clear, in equilibrium intermediaries will enter long positions in asset A and short positions in asset B to satisfy households' hedging demands. Thus, intermediaries will post capital in market B while receiving collateral from market A. Intermediaries can use their received collateral in market B, as the capital market is not segmented. In market A, we assume that households will only need to post collateral to account for the current asset value without including $\bar{\theta}$. Thus, if an intermediary enters a unit of long position in asset A and a short position in asset B, then she has to post collateral equal to

$$P_t^{\mathrm{B}} + \bar{\theta} - P_t^{\mathrm{A}} = \phi_t + \bar{\theta}.$$

The asymmetry in the collateral requirement between the two markets is driven by technical convenience to avoid multiple equilibria in our numerical computation. However, we can also interpret this as capturing the diversity in collateral requirements across different markets.

Thus, we can write the intermediaries' collateral constraints of taking arbitrage size x_t as

$$(1 - m_t)\, k_t^{\mathrm{IM}} Q_t - x_t \left(\phi_t + \bar{\theta}\right) \geq 0.$$

This form of the margin requirement closely follows the prevailing market practice in reality. For OTC derivatives, according to the ISDA Collateral Survey, *"The amount by which the value assigned to the collateral is less than full face value is termed the 'haircut', usually expressed as a percentage of face value"*. As the market value of capital posted as collateral corresponds to the face or notional amount, $(1 - m_t)$ is directly a percentage of the notional amount. Likewise, the haircuts for exchange-traded futures contracts at major futures exchanges such as the Chicago Board of Trade and the Chicago Mercantile Exchange are defined as a specified amount per contract or per notional amount, which is consistent with our setup. In the primary fixed-income markets, for example, repo lending for Treasury, corporate and mortgage-backed securities stipulates that a haircut amounts to a certain ratio of the face amount of the bond. Thus our specification of the margin as a fraction of collateral is also consistent with market practice in the primary fixed-income markets.

In general, by assuming productive capital as collateral, we aim to capture the recent real-world market practice. Although our model is abstract from details of financial innovation or securitization, the collateral with positive haircut rate in our model can be interpreted as securitized agency assets, such as mortgage-backed securities (MBS), credit default swaps (CDS), collateralized debt obligations (CDO), asset-backed securities (ABS), agency bonds, etc. Those securitized products are widely used as collateral in various trading environments, including the markets of CDS, repo and many other OTC derivatives. The corresponding haircut rates reflect the counterparty and market-liquidity risk of these products. These assets are usually very sophisticated and often opaque. Most of them are originated, packed and traded by global investment banks. This is consistent with our assumption that the specialized intermediaries are more efficient than ordinary households in managing the capital.

IV.3 Equilibrium and Aggregation

A competitive equilibrium consists of asset and capital prices P_t^i and Q_t, intermediaries' positions in the risky asset x_t^i and capital allocation k_t^{IM}, and households' position in risky asset y_t^i in market i and capital holdings k_t^i, where $i \in \{\mathrm{A}, \mathrm{B}\}$, such that given prices,

- both intermediaries and households maximize their utility;
- and the markets for risky assets and capital clear:

$$\begin{aligned} &X_t^i + Y_t^i = 0, \\ &K_t^{\mathrm{IM}} + K_t^{\mathrm{A}} + K_t^{\mathrm{B}} = 1, \end{aligned}$$

  where K_t^{IM}, K_t^{A}, K_t^{B} are the aggregate capital holdings in the intermediary sector, market A and market B at t, and X_t^i and Y_t^i are the aggregate positions of intermediaries and households in market i at t.

From the first order condition (IV.6), we can conclude that households' capital holdings in both markets are the same: $k_t^{\mathrm{A}} = k_t^{\mathrm{B}}$ and $K_t^{\mathrm{A}} = K_t^{\mathrm{B}}$, as they share identical preferences and management costs.

Further, because risky assets are in zero net supply and the endowment shocks in market A and B are opposite, the symmetry implies that risk premia are also opposite, i.e. $\psi_t^{\mathrm{A}} = -\psi_t^{\mathrm{B}}$. Thus, the price difference is equal to twice the absolute value of the risk premium, i.e., $\phi_t = P_t^{\mathrm{B}} - P_t^{\mathrm{A}} = 2\psi_t^{\mathrm{A}}$. As the risk premium measures the price divergence between the two assets, its magnitude is also an inverse measure of risky asset market liquidity. When the risk premium is zero, the two assets trade at an equal price and intermediaries provide full liquidity to households across markets. When instead the risk premium is non-zero, liquidity is imperfect.

Intuitively, as pointed out by Gromb and Vayanos (2017), as the intermediaries' positions in risky assets across markets are opposite, their net worth n_t doesn't depend on the dividend d_t of the two assets. Accordingly, risk premia and intermediaries' optimal decisions are also independent of d_t. Since asset dividends are the only source of uncertainty, the price differences and the risk premium are deterministic.

Given the parametrization with which the collateral constraint is binding in equilibrium, we aggregate across intermediaries to obtain the relationship between their total capital value $Q_t K_t^{\mathrm{IM}}$ and their net worth N_t:

$$Q_t K_t^{\mathrm{IM}} = X_t \phi_t + N_t.$$

Summing across both surviving and entering intermediaries yields the evolution of N_t:

$$N_t = \sigma \left[(Z_t + Q_t) K_{t-1}^{\mathrm{IM}} - X_{t-1}\phi_t \right] + W^{\mathrm{IM}}, \tag{IV.9}$$

where $W^{\mathrm{IM}} \equiv (1-\sigma) w^{\mathrm{IM}}$ is the total endowment of new born intermediaries. The first term is the accumulated net worth of intermediaries that operated at $t-1$ and survived to t. It equals to the product of the survival rate σ and the net income over capital and arbitrage $(Z_t + Q_t) K_{t-1}^{\mathrm{IM}} - X_{t-1}\phi_t$. Meanwhile, exiting intermediaries consume the fraction $1-\sigma$ of the total net worth:

$$c_t^{\mathrm{IM}} = (1-\sigma) \left[(Z_t + Q_t) K_{t-1}^{\mathrm{IM}} - X_{t-1}\phi_t \right].$$

IV.3.1 Steady State

If the aggregate productivity multiplier Z_t and the shock intensity u_t are constant, i.e., $Z_t \equiv Z$ and $u_t = u$, then we can derive a steady state of the economy. In particular, we obtain the following relationship reflecting the market liquidity of capital and risky assets.

$$Q = \frac{Z}{r + aK^i(1+r)} = \frac{Z}{r + a\left(1 - K^{\mathrm{IM}}\right)(1+r)/2}, \tag{IV.10}$$

and

$$X = \frac{(1-M)K^{\mathrm{IM}}Q}{\phi + \bar{\theta}} = \frac{(1-M)N}{M\phi + \bar{\theta}}, \tag{IV.11}$$

where K^i, K^{IM}, X, M, Q, N, ϕ are the corresponding steady state variables.

(IV.10) shows that the steady state capital price increases with the proportion of capital held by the intermediary sector. (IV.11) implies that the market liquidity of risky asset increases with intermediaries' net worth while it decreases with the haircut rate in the steady state.

IV.4 Unanticipated Market Freeze

We now consider the possibility of an unanticipated market freeze. Specifically, we use the term "market freeze" to refer to the equilibrium in which intermediaries as a whole are excluded from both financial and capital markets due to insolvency. That is, the entire intermediary sector collapses and hence markets freeze between intermediaries and households. We maintain the assumption that when the counterparty take intermediaries' short positions in period $t-1$, they attach zero probability to a market freeze event at t. Nevertheless, we now allow for the possibility of a freeze ex post when the settlement of the positions is due at t and intermediaries' counterparty must decide whether to continue longing assets from them for another period.

As implied previously, for an intermediary to continue to exploit arbitrage opportunities it must have positive net worth (i.e., $n_t > 0$). Otherwise, it will not fulfill the collateral constraint, which essentially requires intermediaries to use their net worth as part of the collateral. Therefore, it is individually rational for the counterparty to cease longing assets from intermediaries, or provide short-term funding to them, if (i) it perceives that others will also stop buying, forcing intermediaries into a fire sale of the capital and

(ii) this fire sale makes the intermediaries insolvent (i.e., $n_t \leq 0$). In this case, beyond the "normal" equilibrium in which the counterparty keeps trading assets with intermediaries, one other exists: a "freeze" equilibrium. In this equilibrium, the counterparties *en masse* stop renewing their long positions of assets. Intermediaries are forced into liquidation and eventually out of business, as their net worth evaporates. Consequently, households purchase and manage all the capital, while the liquidity of risky assets in segmented markets dries up altogether.

Our modeling of a market freeze as sunspot phenomena is similar to the Gertler and Kiyotaki (2015) model of self-fulfilling bank runs. However, it is not exactly the same. In Gertler and Kiyotaki (2015), intermediaries face a fixed or predetermined amount of debt obligations and a run happens when they are perceived not to have enough net worth to meet the debt requirement at the liquidation price. In contrast, in our model intermediaries' liability (i.e., $x_{t-1}\phi_t$) stems from the mark-to-market settlement of previous arbitrage positions. The amount of this settlement is determined by the prevailing price difference ϕ_t, or risk premium ψ_t. Once the counterparties collectively choose to cease trading with intermediaries, the price difference between segmented markets widens endogenously as the market liquidity dries up, effectively raising intermediaries' liability. Thus, a market freeze equilibrium may exist even if the intermediaries' liquidation of capital at t is sufficient to cover the amount of liability anticipated by both parties at $t-1$. Since intermediaries cannot internalize the price impact of their asset trading volume, in the event of forced liquidation, not only their capital will be worth less, their obligated repayment will also soar.

IV.4.1 Conditions for a Freeze Equilibrium

As in Gertler and Kiyotaki (2015), the market freeze equilibrium occurs when the whole intermediary sector is insolvent. Given the homogeneity of intermediaries in our model, the conditions for a market freeze will be the same for the individual intermediaries' insolvency.

Consider that at the beginning of period t and after the realization of Z_t, the counterparty decides to cease renewing their long positions with intermediaries. The intermediaries then have to liquidate their capital and transfer the proceeds to the counterparty households, who then manage capital in a less efficient manner. Define Q_t^* and ϕ_t^* to be the liquidation capital price and the asset spread. Then a market freeze is likely when the liquidation value of capital, i.e., $(Z_t + Q_t^*)\, K_{t-1}^{\mathrm{IM}}$, is less than intermediaries' due liability, i.e., $X_{t-1}\phi_t^*$. In this case, intermediaries' net worth falls to zero or below. Let l_t be the leverage ratio of $Q_t K_t^{\mathrm{IM}}$ to the net worth N_t, and v_t be the recovery rate in the market freeze as the ratio of $(Z_t + Q_t^*) K_{t-1}^{\mathrm{IM}}$ to $X_{t-1}\phi_t^*$. Then the condition for the market freeze equilibrium to exist is that the recovery rate is less than one:

$$v_t = \frac{(Z_t + Q_t^*) K_{t-1}^{\mathrm{IM}}}{X_{t-1}\phi_t^*} < 1.$$

The condition is determined by three key endogenous factors, the liquidation capital price Q_t^*, the liquidation price spread ϕ_t^* and intermediaries' investment allocation. By rearrangement, we can get a concise form of the freeze equilibrium condition:

$$v_t = R_t^* \frac{l_{t-1}}{l_{t-1} - 1} \frac{\phi_{t-1}}{\phi_t^*} < 1, \tag{IV.12}$$

with

$$R_t^* \equiv \frac{Z_t + Q_t^*}{Q_{t-1}}.$$

where R_t^* is the realized return on capital conditional on the market freezing in period t, and l_{t-1} is the intermediaries' leverage ratio at $t-1$. A market freeze equilibrium exists if the realized return on capital given liquidation R_t^* is sufficiently low, and the realized price spread conditional on liquidation ϕ_t^* as well as the leverage ratio l_{t-1} are sufficiently high to satisfy condition (IV.12). Such condition does not rely on intermediary-specific factors, as $(R_t^*, \phi_t^*, \phi_{t-1}, l_{t-1})$ are identical for all intermediaries in equilibrium.

Figure IV.3 shows how the possibility of a market freeze may hinge on endogenous variables. The vertical axis measures the ratio of realized capital return R_t^* over the risk premium growth ϕ_t^*/ϕ_{t-1} conditional on liquidation. The horizontal axis gauges intermediary's leverage ratios l_{t-1}. The curve which is increasing and concave in $(R_t^*\phi_{t-1}/\phi_t^*,\ l_{t-1})$ space represents combinations for which the recovery rate v_t equals unity. To the left of this curve, intermediaries always stay solvent and the market freeze equilibrium does not exist. To the right, $v_t < 1$ and a market freeze is possible. Similar to Gertler and Kiyotaki (2015), later we conduct simulations with the economy initiating at a point like C in the figure where a market freeze is impossible. A negative shock then raises the asset price spread and reduces the liquidation capital price, shifting the economy to a point like D where the market freeze equilibrium is possible.

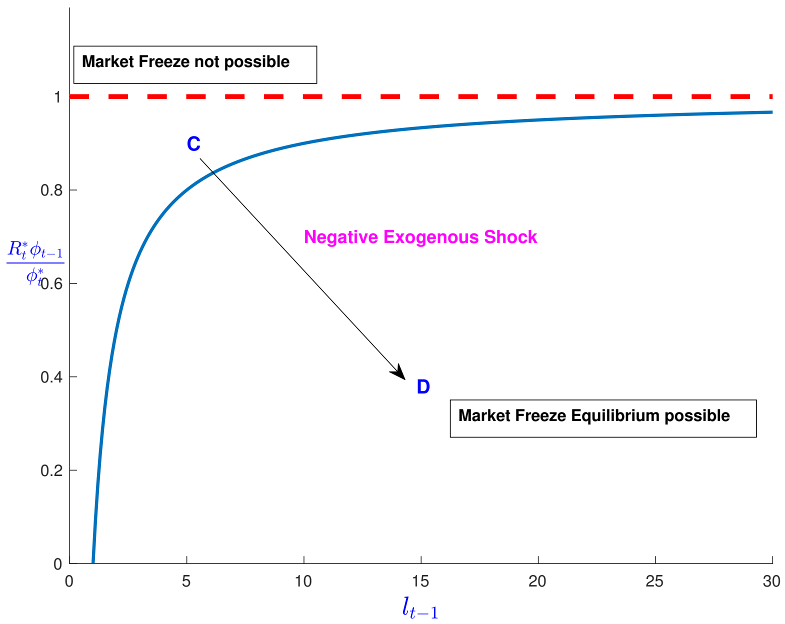

Figure IV.3: Market freeze threshold.

IV.4.2 The Liquidation Capital Price and Risk Premium

In order to determine the liquidation prices, Q_t^* and ϕ_t^*, and to make them quantitatively significant during a market freeze, we adopt Gertler and Kiyotaki (2015)'s assumptions that (i) once a freeze happens, all existing intermediaries liquidate their capital and asset positions, permanently quitting the markets; (ii) the intermediary sector then reconstruct itself over time as new intermediaries enter, and the rebuilding period length is set to be one; (iii) During the reconstruction period, new intermediaries cannot start trading risky assets or acquire capital. One plausible interpretation is that during the freeze it is too costly for the counterparty to identify new intermediaries which are financially independent of those in distress: new intermediaries therefore wait for the air to clear and then engage in arbitrage and manage capital in the following period. The numerical results shown later are robust to alternative timing of the new intermediaries' entry. Given that everything else equal, the severity of the crisis increases with the time it takes for new intermediaries to start operating.

In particular, when intermediaries are forced into liquidation at t, they sell all their capital to households and take no further asset positions, i.e.,

$$
\begin{aligned}
&K_t^{\mathrm{A}} + K_t^{\mathrm{B}} = 1. \\
&X_t = 0.
\end{aligned}
$$

The intermediary sector then starts to accumulate capital and net worth as new intermediaries enter from $t+1$ onwards. Thus, given the above timing assumptions and (IV.9) intermediaries' net worth evolves in the periods following the market freeze according to

$$
\begin{aligned}
&N_{t+1} = w^{\mathrm{IM}} + \sigma w^{\mathrm{IM}}, \\
&N_{t+j} = \sigma[(Z_{t+j} + Q_{t+j})K_{t+j-1}^{\mathrm{IM}} - X_{t+j-1}\phi_{t+j}] + w^{\mathrm{IM}}, \quad \forall j \geq 2.
\end{aligned}
$$

In the period after the market freeze, the aggregate net worth of intermediaries consists of the endowment of newcomers and that of the intermediaries who enter with a delay, supposing that the endowment is durable one-for-one between the periods.

Rearranging the first order condition for the households' capital stocks (IV.6) and financial asset positions (IV.8) yields the following expressions for the liquidation capital price and asset risk premium:

$$
Q_t^* = \mathbb{E}_t\left[\sum_{j=1}^{\infty} \frac{Z_{t+j}}{(1+r)^j \prod_{s=1}^{j}\left(1 + aK_{t+s-1}^i\right)}\right], \tag{IV.13}
$$

$$
\phi_t^* = 2\psi_t^* = 2\bar{\theta}\mathbb{E}_t\left[\sum_{j=0}^{\infty} \frac{g'\left([u_{t+j} - x_{t+j}]\right)}{(1+r)^{j+1}}\right], \quad i \in \{\mathrm{A}, \mathrm{B}\}. \tag{IV.14}
$$

With the proviso that all else equal, the longer it takes for the intermediary industry to fully recover, as measured by the time it takes K_{t+j}^{IM} and X_{t+j} to converge to steady state, the lower will be Q_t^* and the larger will be ϕ_t^*. Note also that the liquidation price and risk premium vary cyclically with productivity. For example, a negative shock to Z_t will decrease Q_t^* and increase ϕ_t^* or ψ_t^*, potentially moving the economy into a regime where a market freeze is possible. On the other hand, as will become clearer later, those price variables also fluctuate countercyclically with the shocks to asset dividend volatility $\bar{\theta}$, with the similar tendency of directing the economy into a state where a market freeze becomes possible.

Comparably, the concept of the liquidation prices in our model corresponds to Brunnermeier and Pedersen (2009)'s notion of market liquidity, while our modeling of margin requirement resonates with their funding liquidity. In our framework and in theirs, the two types of liquidity work jointly in the fire sale panic. Also, the relationship between intermediaries' solvency and market liquidity is similar to that in Gertler and Kiyotaki (2015). When a freeze equilibrium exists, intermediaries become bankrupt, i.e., their liability, when evaluated at ϕ_t^*, exceeds their net worth, when capital is valued at Q_t^*. However, if the price spread and capital are valued at the level in the "normal" equilibrium, i.e., ϕ_t and Q_t, then the intermediaries are all solvent. Hence, the solvency of intermediaries hinges upon the equilibrium prices, which in turn affects and is affected by the funding liquidity of the intermediary sector. Also, both types of liquidity can dry up suddenly in the wake of a market freeze. As a real-world example of this phenomenon, consider the collapse of the hedge funds during the Asia financial crisis. Though some funds took the right direction of arbitrage positions, which would be profitable in the long run, a sudden liquidity shortage escalated into a solvency problem as margin calls lead to price collapse followed by forced liquidation of would-be profitable arbitrage positions and fire sale of long-term capital.

IV.5 Numerical Examples

In this section, we construct two shock scenarios to illustrate how our model works. In particular, in our first example, we let a tiny, temporary negative shock hit the aggregate productivity Z_t at the steady state and then we trace out the effects on financial and real variables. Second, we simulate a small and sudden positive shock in the volatility $\bar{\theta}$ of the asset's dividend and then portray the corresponding impact on both real and financial sectors. After looking at the basic workings of the model, we then allow for an unexpected market freeze ex post and track the subsequent responses. In our examples, the market freeze equilibrium does not exist in the pre-shock state. However, a recession triggered by exogenous shocks might open up the possibility of a sudden market collapse. In general, these numerical exercises aim to study the cyclical responses to macroeconomic

Table IV.1: Parameters

Baseline model		
a	0.01	Households' managmerial cost
g	0.02	Margin requirement coefficient
h	1.02	Margin requirement coefficient
r	0.005	Riskless interest rate
u	16	Households' endowment shock intensity
w^{IM}	0.008	Intermediaries' endowment
Z	0.012	Steady state productivity
α	0.8	Households' coefficient of risk aversion
β	0.99	Households' discount rate
ρ	0.9	Serial correlation of productivity shock
σ	0.95	Intermediaries' survival probability
$\bar{\theta}$	0.004	Upper bound of endowment shock unit

Table IV.2: Steady State Values

Baseline model	
K	1
K^{IM}	0.5906
K^{A}	0.2047
K^{B}	0.0472
Q	0.4835
X	1.9283
Y	0.0103
N	0.1467
n	0.1540
ϕ	0.0720
ψ	0.0360
m	0.4864

and financial shocks as well as to illustrate the real effects of the market freeze. Given the abstract nature of our model, these numerical examples are not accurate estimates.

IV.5.1 Parameter Choices and Computation

Table IV.1 lists the choice of parameter values for our baseline model, while Table IV.2 presents the steady state values of the endogenous variables. We take the time interval to be one month. Overall, there are eleven key parameters in the baseline model. Except for the serial correlation ρ of the productivity shock Z_t, the remaining ten parameters (α, σ, $\bar{\theta}$, a, g, h, r, u, w^{IM}, Z) are specific to our model.

As we noted earlier, the intermediaries in our model correspond best to the speculating institutions in the shadow banking system, such as hedge funds or other market makers, who tend to operate with rather moderate leverage ratios. Also, we aim to capture the real world phenomenon that there are unabsorbed arbitrage opportunities or persistent price differences between similar assets in different markets or subsectors. Therefore, we choose values for the riskless interest rate r, households' aggregate size of asset demand u, their coefficient of risk aversion α, the endowment shock unit magnitude $\bar{\theta}$, intermediary's initial endowment w^{IM} and margin requirement parameters g and h to satisfy the following targets in the steady state: an intermediary's leverage ratio is below two and its collateral constraint is binding. It is hard to gather precise income statements or capital flows for the entire intermediary industry. Thus, the values we choose here are intended to be reasonable benchmarks that reflect the fragility of the financial positions of the speculating institutions. The results are robust to plausible variations, such as higher leverage ratios and more risk averse intermediaries.

In addition, we set the intermediary's survival probability σ equal to 0.95, which implies an expected horizon of twenty months. Also, we set the parameter that reflects "managerial cost" a to 0.1, a value low enough to ensure that households find it profitable to directly hold capital, but high enough to produce a reasonable degree of decrease in the capital price in the event of the intermediaries' financial distress.

IV.5.2 Amplification and Spillover Effects Initiated from Production Sector

Figure IV.4 shows the response of the baseline model to an unanticipated negative 2.8 percent shock to productivity Z_t. We choose the size of shock in order to generate a fall in output close to the one that occurred during the Great Recession. The shock leads to a drop in aggregate output (total output minus management costs) up to 8 percent, a magnitude that matches the data during the Great Recession. The recession induces financial distress that amplifies the fall in the capital price and raises the haircut rate in the financial markets. In particular, the unexpected 2.8 percent fall in productivity Z_t reduces the capital price Q_t by over 6 percent and pushes up the risk premium ψ_t by nearly 0.4 percent. The depressed capital price further tightens the haircut m_t by about 6 percent. Meanwhile, the sudden negative shock sharply reduces the intermediaries' net worth N_t roughly by 12 percent, causing a slump in the arbitrage positions up to 21 percent and a fire sale of 11 percent of the capital.

Overall, as illustrated by Figure IV.5, the recession induces two different feedback loops, which reinforce each other and give rise to a more complicated financial accelerator

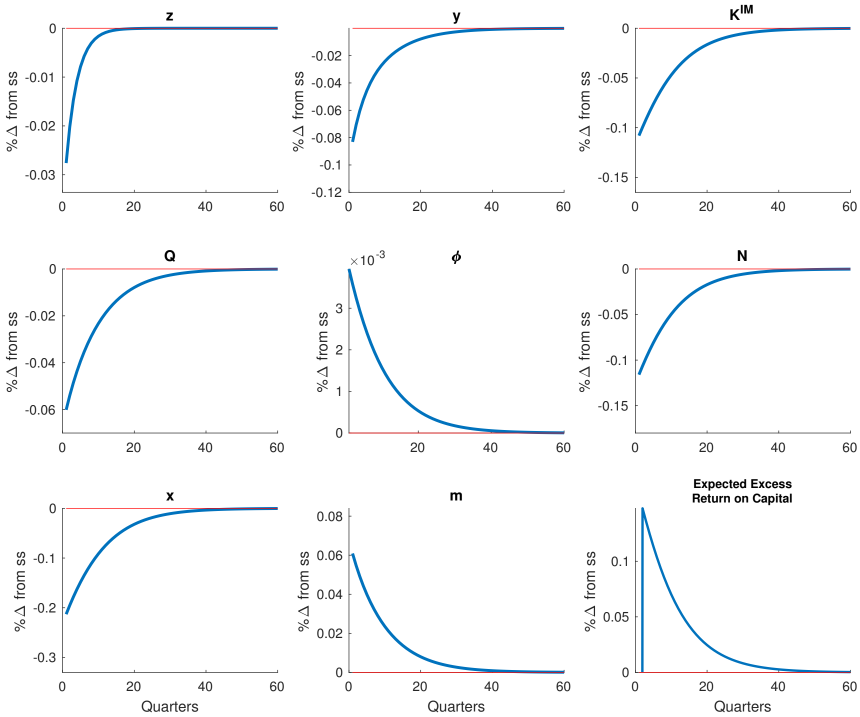

Figure IV.4: A recession in the baseline model.

mechanism than the ones prevalent in Bernanke and Gertler (1989) and Kiyotaki and Moore (1997). One feedback loop is similar to the above mentioned literature. The recession causes financial distress that amplifies the fall in capital prices and raises the cost of intermediaries' arbitrage trading. The unanticipated drop in Z_t reduces the production income $Z_t K_{t-1}^{\mathrm{IM}}$ and hence net worth N_t. This tightens their budgets and leads to a fire sale of capital, which in turn magnifies the decline of the capital price Q_t. Households purchase some of the capital. But as it is increasingly costly for them to manage capital, the amount they absorb is limited. As a result, the net effect is a sharp increase in the spread between the expected return to capital and the riskless rate.

The other feedback loop is specific to our model. The fall in capital price Q_t raises the intermediaries' funding cost through the rise in the margin m_t. This not only reduces the capital worth, but also tightens the margin requirement. Thus, intermediaries no longer can maintain their previous level of liquidity supply in the financial markets, even if their capital were not depreciated in value. Since they cannot internalize the price effects of their collective liquidity supply, when they unwind part of their positions, the price spread widens and this incurs losses in their previous positions. Consequently, the losses further fuel the fire sale of capital and the price decline. Similar to the mechanism

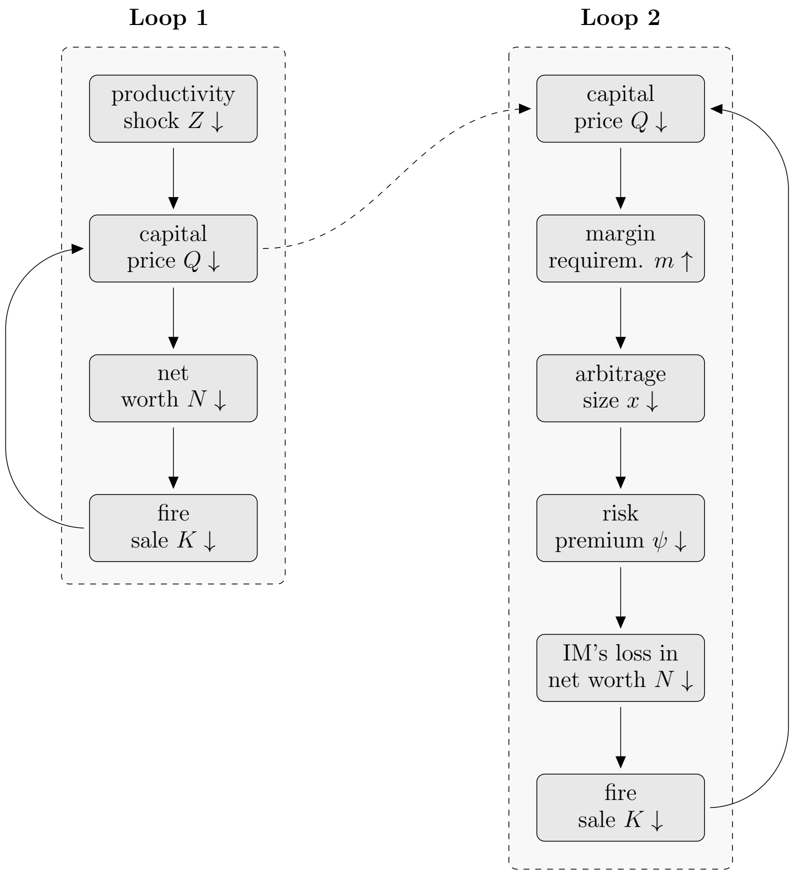

Figure IV.5: The structure of the economic system.

in Brunnermeier and Pedersen (2009) and Gromb and Vayanos (2017), the intermediaries' funding capability gets damaged the most when their arbitrage profitability is the highest.

Thus, in this sense, the negative shock in the production sector spills over to financial markets, causing losses in intermediaries' net worth. This in turn deepens the distress in the real economy: it exacerbates the fire sale of capital and asset misallocation. The loss spiral reinforces the market liquidity spirals for both risky asset and capital, together contributing to a sharp contraction in the aggregate activities.

IV.5.3 Amplification and Spillover Effects Triggered from Financial Markets

Figure IV.6 exhibits the response of the baseline model to an unanticipated positive 4 percent shock to the volatility measure of asset dividends or the upper bound of household endowment shock unit $\bar{\theta}$. We assume that $\bar{\theta}$ also follows an AR(1) process with serial correlation $\rho_{\bar{\theta}} = 0.96$. This leads to a drop in aggregate output of over 2 percent, which

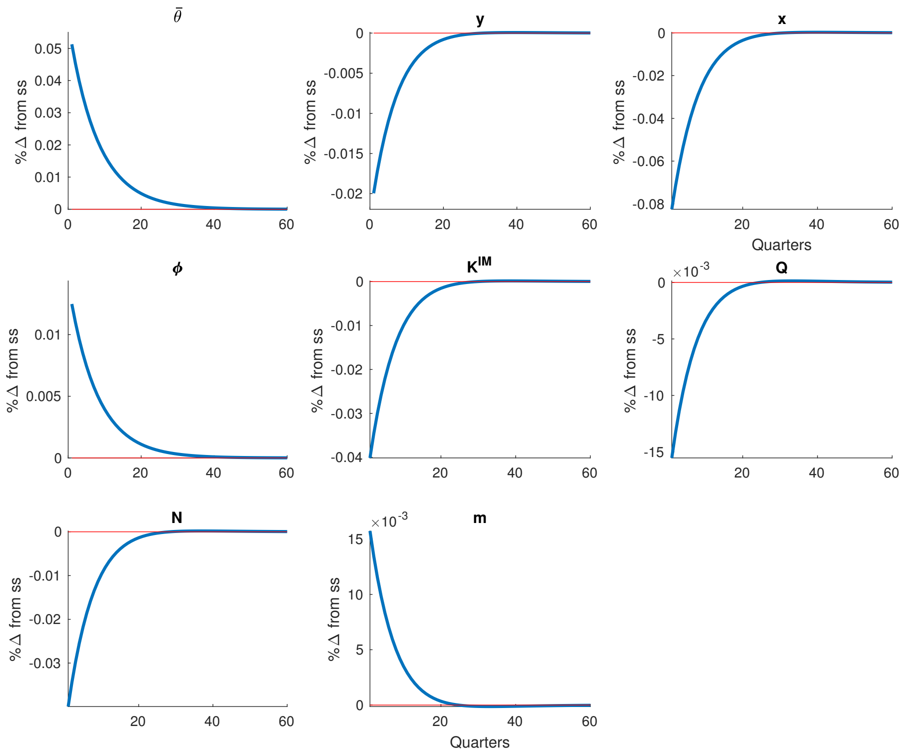

Figure IV.6: A recession in the baseline model triggered by a shock in financial markets.

amounts to a medium-sized recession in the real economy, though there is no change to productivity itself and households' wealth effect is isolated. The shock in the payoff of financial assets increases the asset specific margin requirement, which raises intermediaries' funding costs and triggers the spillover effects to the real economy. The unexpected shock in $\bar{\theta}$ reduces intermediaries' net worth n_t by roughly 4 percent, which tightens their budget and triggers a fire sale of capital and the contraction of arbitrage volume. In particular, intermediaries' capital holding drops by 4 percent, the market liquidity slumps by about 8 percent, and the risk premium rallies by almost 1.3 percent.

As illustrated by Figure IV.7, the spillover and amplification effects arise from the intermediaries' increasing funding illiquidity. Though a surge in asset dividend volatility helps increase their arbitrage profitability, the more dominating effect is that it also significantly raises the asset specific margin, hence suppressing their funding capability through arbitrage. Thus, despite the increasing profitability, intermediaries cannot afford to maintain the pre-shock capital size and liquidity supply. As they cannot internalize the impact of their unwinding decisions on the risk premium, they suffer further losses from increasing liability of their previous positions. The unexpected losses with less renewed funding from arbitrage lead to a fire sale of capital, which depresses the capital prices and

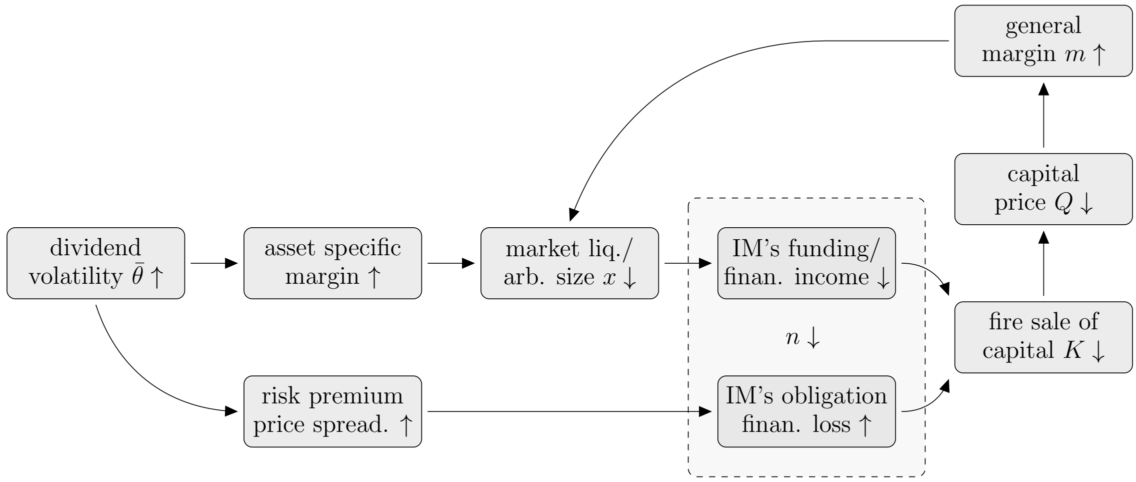

Figure IV.7: The transmission channel of shocks from financial markets

further raises intermediaries' funding cost through an even tighter margin requirement m_t (up to 1.5 percent in our example). In a nutshell, the subsequent decline in funding liquidity not only results in magnified evaporation of market liquidity in asset markets, but also spills over to the real economy by aggravating the capital misallocation.

IV.5.4 Unanticipated Market Freeze

Figure IV.8 presents the response of a productivity shock with an ex post sudden market freeze. As we mentioned before, a freeze equilibrium exists if the recovery rate v_t is less than one. Thus, denote the variable FREEZE_t as the indicator variable that accounts for the deficit of the recovery rate below one:

$$\text{FREEZE}_t \equiv 1 - v_t.$$

A freeze equilibrium exists if and only if $\text{FREEZE}_t > 0$. The first panel of the last row shows that the freeze indicator turns above zero upon the shock and stays positive for some periods. An unexpected market collapse is thus likely to happen at any period in this interval. The direct trigger is that the negative productivity shock reduces the liquidation price Q_t^* and hence lowers the intermediaries' net worth. Meanwhile, the price decline leads to the rise of the haircut rate, further damaging intermediaries' funding capacity. Indirectly, this widens the price spread ϕ_t^* and raises intermediaries' liability. Both effects work to increase intermediaries' leverage ratio ex post and make the whole

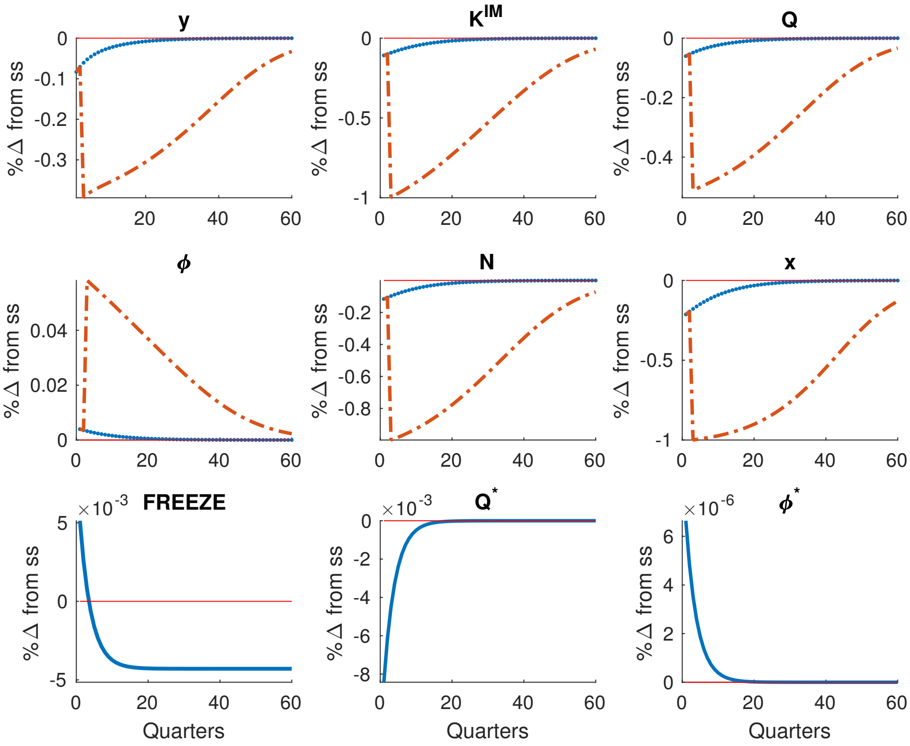

Figure IV.8: Ex post market freeze.

sector susceptible to a sudden market freeze, as implied by (IV.12). In our example, the freeze indicator of the steady state is negative, which implies that the market freeze equilibrium is not feasible in its neighborhood.

In Figure IV.8 we let an unexpected freeze happen half a year after the shock. The blue densely dotted line traces the no-freeze equilibrium and the red dashed line describes the path with a sudden freeze. The market freeze leads to complete liquidation of intermediaries' capital and asset positions as K_t^{IM} and X_t fall to zero in period 6 following the shock. The capital price accordingly drops to its liquidation price at nearly 50 percent below the steady state. Meanwhile, the asset price spread soars to its liquidation level at around 6 percent above the pre-shock state. Total production output net of management costs falls by about 40 percent. Higher management costs arise because of the extremely inefficient capital allocation after the collapse. Intermediaries' net worth also slumps almost to zero as existing intermediaries are completely out of business and newcomers take a very long time to recapitalize.

From the seventh period onward, new intermediaries enter and the intermediary sector starts to rebuild itself. Unlike Gertler and Kiyotaki (2015)'s bank run case, due to the high funding cost arising from the low capital price, intermediaries initially recover in a very slow pace despite the high arbitrage profitability and production return. The recovery speed hinges on how fast intermediaries can accumulate their net worth, as low net worth turns to be associated with low leverage ratio in our model.

IV.6 Intervention Policy Experiments

In this section, we quantify the effects of three types of ex-post policy interventions in the market freeze of the model: (i) direct purchase of capital by the government; (ii) government acting as the lender of last resort by infusing equity to intermediaries or providing loans; (iii) lowering the interest rate. We choose these policies because they are among the ones undertaken by central banks in reality. Our goal is to study the effects of these policies on recovery based on our model. We are not concerned with the potential costs associated with implementing these interventions.

Our ex-post policy experiments correspond to the following exercise. Suppose we are in a state where the old intermediaries are already wiped out and the newly formed intermediary sector is about to operate. From this initial condition, suppose that the central bank implements a policy that was not anticipated by the agents. We track the effects of this policy on the recovery of the economy from this state. Specifically, we compute a new with-policy equilibrium and compare it with the one without intervention, which we have studied in Section IV.5.4.

IV.6.1 Direct Capital Purchase

During both the subprime crisis and the Great Depression, central banks entered the capital markets and conducted a series of direct purchases of the distressed assets, which were held mainly by intermediaries. For example, following the collapse of the intermediary sector, the Federal Reserve initiated the large scale asset purchases (LSAPs) of high grade long term debt, including primarily agency mortgage backed securities (AMBS). From August 2007 to August 2009, around USD 1.8 trillion was injected by the Fed and the US government to the mortgage-backed securities market. By acting as buyers in the capital market, the goal is to raise the capital price and increase the liquidity in the financial sector.

To evaluate the effects of this policy, we conduct a similar experiment as in Gertler and Karadi (2011); Gertler et al. (2016). Specifically, we assume that the central bank can directly manage the capital. Like the household investors, the central bank also bears quadratic managerial costs $\frac{1}{2}Q_t a^g \left(K_t^g\right)^2$, where K_t^g is the size of the central bank's purchase and $a^h > a^g$. To discourage the central banks from holding capital in a non-freeze time or to only allow them to purchase capital upon crises, we further introduce certain degrees of inefficiency in their overall capital return. That is, we impose a discount factor $\chi \in (0, 1)$ on their capital return:

$$R_{t+1}^g = \chi \frac{Z_{t+1} + Q_{t+1}}{Q_t \left(1 + a^g K_t^g\right)}.$$

We assume that the central bank enters the capital market once the expected capital return exceeds its borrowing costs, which is $1 + r$ in our model. Accordingly, the policy rule for central bank's capital purchase is

$$\begin{aligned} &K_t^g = 0, && \text{if } \mathbb{E}\left[R_{t+1}^g\right] - (1 + r) < 0, \\ &\mathbb{E}\left[R_{t+1}^g\right] - (1 + r) = 0, && \text{if } K_t^g \geq 0. \end{aligned}$$

We choose the value of χ to ensure that the central bank only intervenes after a market freeze happens. We also select the management cost parameter a^g such that the amount of government purchase is around 10 percent of the total capital supply.

Figure IV.9 shows the effect of the government capital purchase when a market freeze happens six months after the initial productivity shock. The total purchase is around 10 percent of the aggregate capital. It dampens the drop in the capital price and the aggregate output by around 4.9 percent and 5.5 percent respectively. However, after this initial improvement, the following recovery path of the economy is slower than the one without intervention. Intuitively, the purchase has no further stimulating effect because though the higher capital price helps alleviate intermediaries' funding difficulty, it also raises their cost of acquiring capital. In effect, the government purchase does not only crowd out household investors, but also crowds out intermediaries by lowering their overall capital return, giving rise to a slower recovery.

IV.6.2 Capital Injections

Another type of common interventions aims at directly increasing the intermediaries' ability to purchase capital, either through direct lending or equity injection. For instance, during the subprime crisis, the US Treasury acquired USD 205 billion of preferred shares

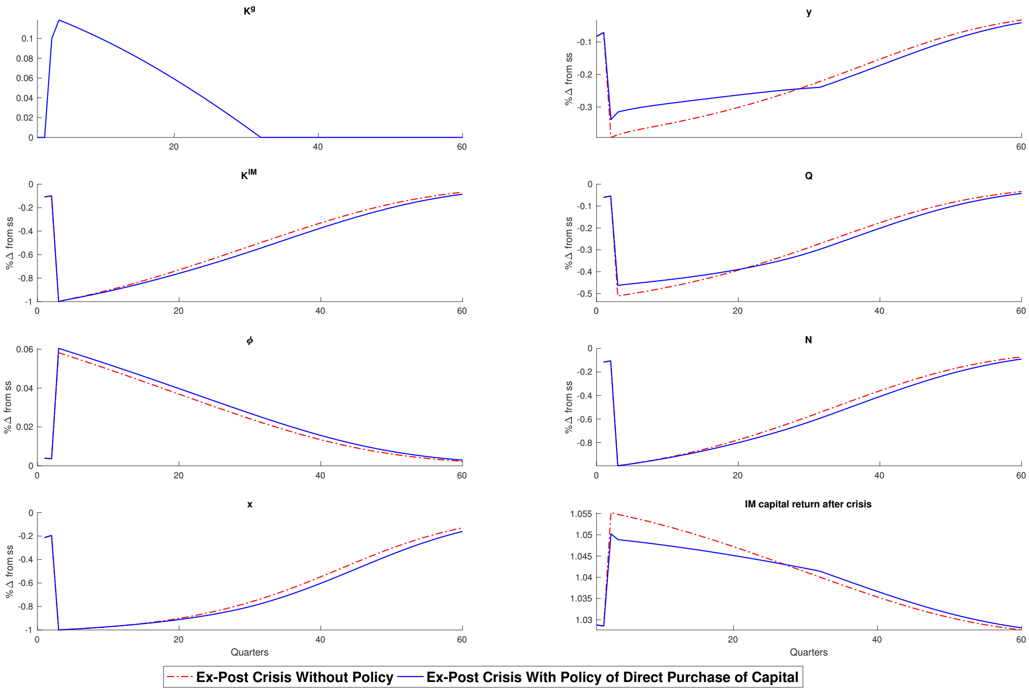

Figure IV.9: Government direct purchase of capital after a crisis.

in the intermediary sector through the capital purchase program. The treasury's capital injection was distributed across many institutions, from pure lending banks to trading firms. Also, in the wake of market collapse, the Fed established special programs such as Primary Dealer Credit Facility (PDCF) and Commercial Paper Funding Facility (CPFF) to provide loans to primary broker-dealers and investment vehicles.

We focus our attention on the case of equity injections, in which the government acquires ownership shares in intermediaries. To model this type of intervention, we assume that the central bank acquires equity from the newly born intermediaries after the market freeze. The central bank may hold the equity stake until the intermediaries exit and then receive the liquidation value of its net worth, equal to their share of $(Q_{t+1} + Z_{t+1})\, k_t^{\mathrm{IM}} - x_t\phi_{t+1}$. Or it may sell off its holding at the market price before the intermediaries exit, when the market recovers to a satisfying level. As our aim is to study and compare the recovery paths of the economy, we assume that the central bank will inject the same amount of resources as the maximal level in the previous direct purchase experiment. Since we do not explicitly model the agency cost, as long as it fulfills the intermediaries' incentive constraints, the specific price at which the central bank acquires the equity relative to the market price is irrelevant to our experiment. For simplicity, in our experiment, we assume that the central bank holds the equity until the intermediaries quit.

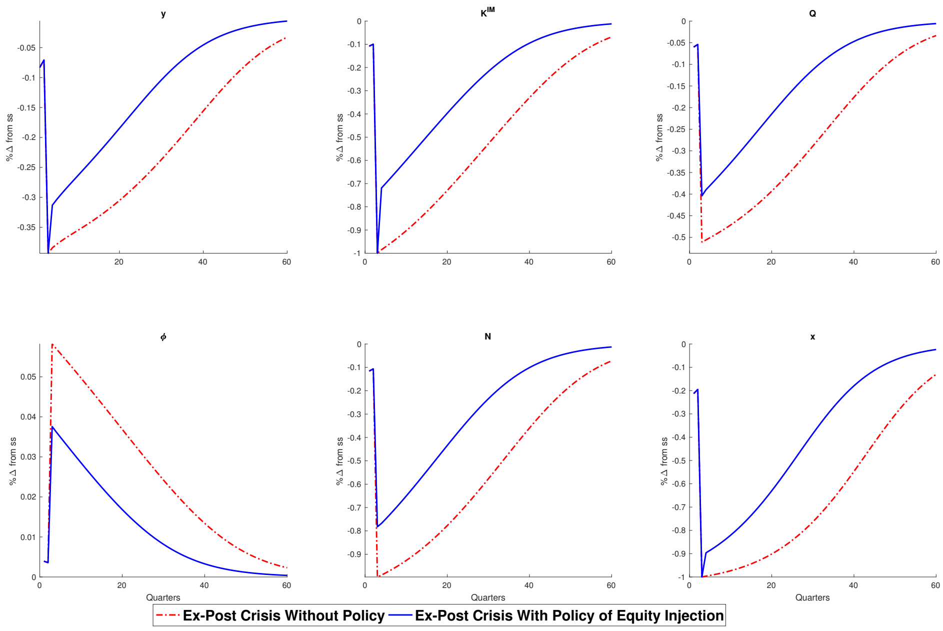

Figure IV.10: Government injection of equity after a crisis.

Figure IV.10 shows the effects of the equity injection after the market freeze six months following the initial productivity shock. The injection dampens the capital price and output drop by around 11 and 7 percent respectively. Also it helps narrow the asset spread by 2 percent. Moreover, this policy significantly speeds up the recovery of the market liquidity as well as intermediaries' recapitalization process. The simulation from the policy of direct lending to intermediaries yields quantitatively similar results. This kind of policy has better effect since part of the friction in our model stems from the constraint on intermediaries' net worth. Both injecting equity and direct lending help intermediaries quickly restore their net worth.

IV.6.3 Interest Rate Cut

Following the onset of the recent crisis, the central bank lowered its discount rate and its target for the overnight interbank interest rate, while expanding the range of assets that qualify as collateral. To the extent that intermediaries rely heavily on rolling over the external funding for their operations, lowering interest rate aims to ease the funding cost and mitigate the stress of forced fire sales. For example, between September 2007 and May 2008, the Fed lowered its target for the Federal funds rate from 5.25% to 2% and the discount rate from 5.75% to 2.25% through six separate actions.

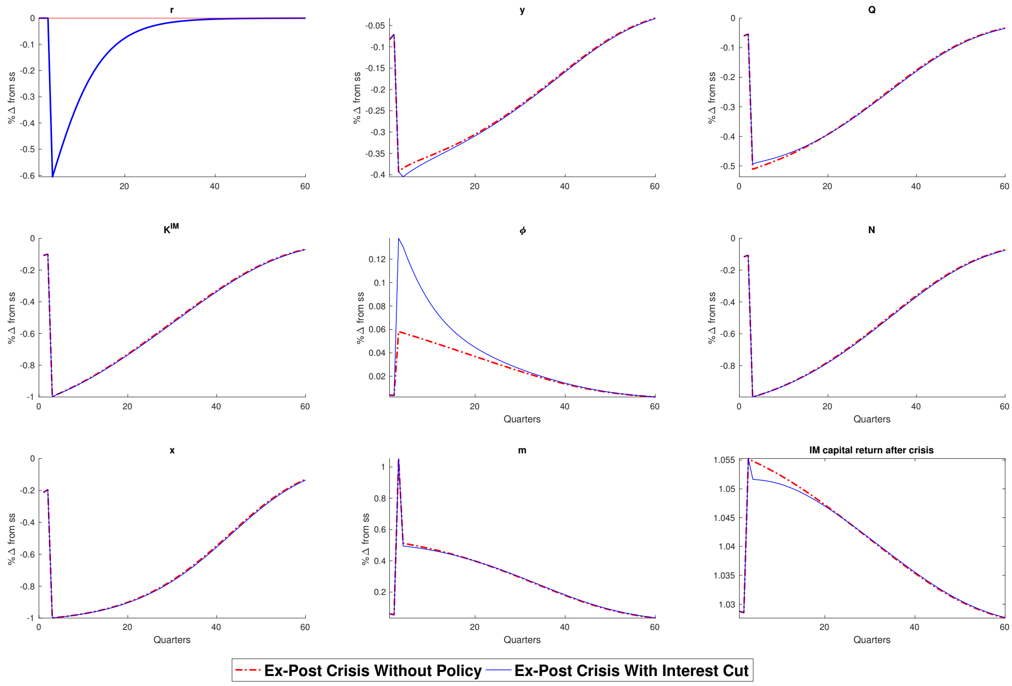

Figure IV.11: Interest rate cut after a crisis.

To study the ex-post intervention of the sharp interest cut, we assume that the central bank lowers the interest rate r after the market freezes six months following the initial shocks in aggregate productivity Z. For simplicity, we model the cut in interest rate as a 60% decline with an autoregressive factor of 0.95.

Figure IV.11 compares the effect of an interest cut with the policy-free one. The intervention leads to an upward jump in the risk premium and the capital price. The lower interest rate significantly raises households' demand for the risky asset as well as the capital. In turn, this implies that intermediaries now face higher marginal cost of capital, and accordingly their demand of capital falls. Effectively, this policy redirects capital from intermediaries to households, thereby reducing the overall efficiency and output level. After the initial few periods, the recovery path is nearly the same as in the case without intervention. Intuitively, the interest rate cut has an overall detrimental effect on recovery, as intermediaries' improved arbitrage profitability and better funding liquidity are offset by their diminished return of capital. Similar to the direct purchase case, the intermediaries are crowded out from the capital market by the households. Therefore, less overall return accrues to their net worth, leading to a slower recovery.

IV.7 Conclusion

We have developed a model to study the interaction of intermediation in both financial and real sectors and its effects on limits of arbitrage. By linking the market liquidity of productive capital to arbitrageurs' margin requirements in financial markets, we show that market liquidity across the fundamentally independent asset markets co-move, leading to correlated limits of arbitrage. In doing so, we provide an alternative channel to model the financial accelerator effects and self-fulfilling market collapse. We demonstrate that intermediaries' arbitrage losses reinforce the deterioration of intermediation in both sectors. Also, we illustrate that, while in normal times a self-fulfilling market freeze equilibrium might not exist, the possibility can arise during an economic downturn. This can occur even to intermediaries with a very modest degree of leverage.

There are many possible extensions of this model. For example, we can allow for more uncertainties in asset demands in the financial markets, in which arbitrageurs have to develop non-trivial optimal trading strategies. We can then examine the effects on the price volatility and the risk premia. Also, by adding more distinct financial securities and segmented markets, we can investigate the cross-sectional dynamics and the implications of capital mobility.

One limitation of our model is that it does not fully endogenize the collateral constraints. Potential improvement can be made by incorporating moral hazard, or by allowing for a complete market of collateral contracts. Also, by fixing the total supply of capital, the model sheds no light on how the intermediation performance affects the economic growth. Including capital accumulation and the process of securitization will help us conduct the welfare analysis and evaluate the effects of policies in a more precise and comprehensive way.

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IV.A Proofs

IV.A.1 Proof of Proposition IV.1

Proof. From the first order condition in (IV.7)

$$\mathbb{E}_t\left[\Gamma_{t,t+1}\frac{P^i_{t+1}+d_{t+1}}{P^i_t}\right]=1.$$

and

$$\begin{aligned}P^i_t &= \frac{\bar{d}}{r}-\psi^i_t,\\ P^i_{t+1} &= \frac{\bar{d}}{r}-\psi^i_{t+1}\end{aligned}$$

we have

$$\begin{aligned}&\mathbb{E}_t\left[\Gamma_{t,t+1}\frac{\bar{d}/r-\psi^i_{t+1}+\bar{d}+\theta_{t+1}}{\bar{d}/r-\psi^i_t}\right]\\ &=\mathbb{E}_t\left[\Gamma_{t,t+1}\frac{(1+r)\bar{d}-r\psi^i_{t+1}}{\bar{d}-r\psi^i_t}\right]+\mathbb{E}_t\left[\Gamma_{t,t+1}\frac{r\theta_{t+1}}{\bar{d}-r\psi^i_t}\right]=1.\end{aligned}$$

As

$$R\equiv\frac{(1+r)\bar{d}-r\psi^i_{t+1}}{\bar{d}-r\psi^i_t}$$

is deterministic and using (IV.4) we obtain

$$\mathbb{E}\left[\Gamma_{t,t+1}\theta_{t+1}\right]=\frac{\psi^i_{t+1}}{1+r}-\psi^i_t=-\frac{\Phi^i_t}{1+r}.$$

On the other hand, from the definition of (IV.5), we have

$$\begin{aligned}\Gamma_{t,t+1} &= \beta\exp\left\{-\alpha\left(c^i_{t+1}-c^i_t\right)\right\}\\ &= \beta\exp\left\{-\alpha\left(S_{t+1}-c^i_t+\left(y^i_t+u^i_t\right)\theta_{t+1}\right)\right\},\end{aligned}$$

where

$$
\begin{aligned}
S_{t+1} \equiv\; & y_t^i \left(\bar{d} + P_{t+1}^i\right) + \left(Z_{t+1} + Q_{t+1}\right) K_t^i + (1+r) B_t^i - y_{t+1}^i P_{t+1}^i - Q_{t+1} K_{t+1}^i \\
& - \frac{a}{2} Q_{t+1} \left(K_{t+1}^i\right)^2 - B_{t+1}^i
\end{aligned}
$$

is the deterministic part of the c_{t+1}^i. Thus, combining Equations (IV.4) and (IV.A.1), we can derive

$$
\frac{\mathbb{E}\left[\theta_{t+1} \exp\left\{-\alpha \left(y_t^i + u_t^i\right) \theta_{t+1}\right\}\right]}{\mathbb{E}\left[\exp\left\{-\alpha \left(y_t^i + u_t^i\right) \theta_{t+1}\right\}\right]} = -\Phi_t^i. \tag{IV.15}
$$

Define a function $g(x)$ as

$$
\exp\{\alpha g(x)\} = \mathbb{E}\left[\exp\left\{-\frac{\alpha x \theta_t}{\bar{\theta}}\right\}\right],
$$

then

$$
\mathbb{E}\left[\exp\left\{-\alpha \left(y_t^i + u_t^i\right) \theta_{t+1}\right\}\right] = \exp\left\{\alpha g\left(\left(y_t^i + u_t^i\right) \bar{\theta}\right)\right\}. \tag{IV.16}
$$

Thus, if we take first order derivatives with respect to y_t^i of (IV.16), we can obtain

$$
-\alpha \mathbb{E}\left[\exp\left\{-\alpha \left(y_t^i + u_t^i\right) \theta_{t+1}\right\} \theta_{t+1}\right] = \alpha \bar{\theta} g'\left(\left(y_t^i + u_t^i\right) \bar{\theta}\right) \mathbb{E}\left[\exp\left\{-\alpha \left(y_t^i + u_t^i\right) \theta_{t+1}\right\}\right]. \tag{IV.17}
$$

Combining with (IV.15) and simplifying, we get

$$
g'\left(\left(y_t^i + u_t^i\right) \bar{\theta}\right) \bar{\theta} = \Phi_t^i. \tag{IV.18}
$$

The proof of the properties of function $g(x)$ is shown in Gromb and Vayanos (2017), we rewrite it here with minor adjustments to adapt to our notations.

For expositional convenience, we set $\hat{\theta}_t \equiv \theta_t / \bar{\theta}$. Since the distribution of $\hat{\theta}_t$ is independent of t, so if the function $g(x)$. As $\hat{\theta}_t$ has mean zero, Jensen's inequality implies that

$$
\mathbb{E}\left[\exp\left\{-\alpha x \hat{\theta}_t\right\}\right] \geq 1,
$$

and thus $g(x) \geq 0$. Because $\hat{\theta}_t$ is symmetric around zero,

$$\mathbb{E}\left[\exp\left\{-\alpha x\hat{\theta}_t\right\}\right] = \mathbb{E}\left[\exp\left\{\alpha x\hat{\theta}_t\right\}\right],$$

$g(x)$ is symmetric around the vertical axis.

To show that $g(x)$ is strictly convex, we prove that $g''(x) > 0$ in the following. As

$$g(x) = \frac{1}{\alpha}\log\left(\mathbb{E}\left[\exp\left\{-\alpha x\hat{\theta}_t\right\}\right]\right),$$

differentiating it, we get

$$g'(x) = -\frac{\mathbb{E}\left[\hat{\theta}_t \exp\left\{-\alpha x\hat{\theta}_t\right\}\right]}{\mathbb{E}\left[\exp\left\{-\alpha x\hat{\theta}_t\right\}\right]}.$$

Differentiating it again, we obtain

$$g''(x) = \alpha\frac{\mathbb{E}\left[\hat{\theta}_t^2 \exp\left\{-\alpha x\hat{\theta}_t\right\}\right]\mathbb{E}\left[\exp\left\{-\alpha x\hat{\theta}_t\right\}\right] - \left(\mathbb{E}\left[\hat{\theta}_t \exp\left\{-\alpha x\hat{\theta}_t\right\}\right]\right)^2}{\left(\mathbb{E}\left[\hat{\theta}_t \exp\left\{-\alpha x\hat{\theta}_t\right\}\right]\right)^2}. \tag{IV.19}$$

The numerator in (IV.19) is positive because of the Cauchy-Schwarz inequality $(mathbbE[ST])^2 \leq \mathbb{E}\left[S^2\right]\mathbb{E}\left[T^2\right]$, which is strict when the random variables S and T are not proportional. Thus, we can apply the Cauchy-Schwarz inequality by setting

$$\begin{aligned} S &\equiv \hat{\theta}_t \exp\left\{-\frac{\alpha x\hat{\theta}_t}{2}\right\}, \\ T &\equiv \exp\left\{-\frac{\alpha x\hat{\theta}_t}{2}\right\}, \end{aligned}$$

and S and T are not proportional as $\hat{\theta}_t$ is stochastic. Hence, $g''(x) > 0$.

To prove that $\lim_{x\to\infty} g'(x) = 1$, one can show that $|g'(x) - 1|$ can be made smaller than 2η for any arbitrary $\eta > 0$ when $x \to \infty$. Following Equations (IV.A.1) and $\hat{\theta}_t$ is symmetrically distributed around zero with the supremum of its support being one,

$$
\begin{aligned}
\left|g'(x) - 1\right| &= \frac{\left(1+\hat{\theta}_t\right)\mathbb{E}\left[\exp\left\{-\alpha x\hat{\theta}_t)\right\}\right]}{\mathbb{E}\left[\exp\left\{-\alpha x\hat{\theta}_t\right\}\right]} \\
&= \frac{\left(1+\hat{\theta}_t\right)\mathbb{E}\left[\exp\left\{-\alpha x\hat{\theta}_t\right\}\mathbf{1}_{\{\hat{\theta}_t\in[-1,-1+\eta]\}}\right]}{\mathbb{E}\left[\exp\left\{-\alpha x\hat{\theta}_t\right\}\right]} \\
&\quad + \frac{\left(1+\hat{\theta}_t\right)\mathbb{E}\left[\exp\left\{-\alpha x\hat{\theta}_t\right\}\mathbf{1}_{\{\hat{\theta}_t\in[-1+\eta,1]\}}\right]}{\mathbb{E}\left[\exp\left\{-\alpha x\hat{\theta}_t\right\}\right]}.
\end{aligned}
\tag{IV.20}
$$

As

$$
\left(1+\hat{\theta}_t\right)\mathbf{1}_{\{\hat{\theta}_t\in[-1,-1+\eta]\}} \leq \eta,
$$

the first term in the right hand side of (IV.20) is less than η. The second term can also be less than η given large enough x. Multiplying numerator and denominator by $\exp\{-\alpha x(1-\eta)\}$, one can rewrite the term as

$$
\frac{\mathbb{E}\left[\left(1+\hat{\theta}_t\right)\exp\left\{-\alpha x\left(\hat{\theta}_t+1-\eta\right)\right\}\mathbf{1}_{\{\hat{\theta}_t\in[-1+\eta,1]\}}\right]}{\mathbb{E}\left[\exp\left\{-\alpha x\left(\hat{\theta}_t+1-\eta\right)\right\}\right]}.
\tag{IV.21}
$$

Since $\hat{\theta}_t$ in the numerator in (IV.21) is greater than $-1+\eta$, the numerator remains bounded when $x \to \infty$. Yet, the denominator in (IV.21) converges to infinity, as $\hat{\theta}_t$ takes value in $[-1, -1+\eta)$ with positive probability. $\square$

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